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KEY,

CONTAINING

SOLUTIONS AND EXPLANATIONS

OF THE PROBLEMS IN

THE LAST FOUR BOOKS

OF THE

ELEMENTS OF GEOMETRY;

BY BENJAMIN GREENLEAF, A.M.

AUTHOR OF A MATHEMATICAL SERIES.

FOR THE USE OF TEACHERS ONLY.

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Having carefully read the advanced sheets of Greenleaf's Geometry, I am happy to be able to assure you that the work appears to me to be one of great merit. Of course, no person can expect to find much that is essentially new in a geometrical treatise published at this day. But Mr. Greenleaf's, while it is based, like nearly all American Geometries, upon Legendre's original work, contains valuable propositions not usually found in similar publications, and also presents uncommonly clear demonstrations of many of the standard propositions. The definitions contained in this work are generally consise, clear, and exact. The exercises in Mensuration, and in the Application of Algebra to Geometry, will be found unusually interesting and useful. In a word, I have no doubt that Mr. Greenleaf's Geometry will be found to compare favorably with the best Geometries yet published.

D. B. HAGAR,
Principal of Eliot High School.

WEST ROXBURY, Mass., July 3, 1858.

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KEY

TO

GREENLEAF'S GEOMETRY.

BOOK XI.

PROBLEM I. — PAGES 254, 255.

3. Ans. $462\frac{1}{4}$ sq. ft.

4. Ans. 124 A. 1 R. 1 P.

6. Ans. 27 sq. ft.

8. $68\frac{1}{4} \times 56\frac{2}{3} = 3867\frac{1}{2}$; $3867\frac{1}{2} \div \frac{15}{8} = 2062\frac{2}{3}$.

10. Ans. 96 sq. ft.

11. $1234 \times 658 = 811972$ = no. sq. links in the field; $811972 \div 100000 = 8.11972$ = no. acres; 8.11972 A. = 8 A. 19 P. 4 yd. $6\frac{1}{4}$ ft., nearly.

PROBLEM II. — PAGES 255, 256.

2. Ans. 3525 links.

3. 15 A. 2 R. 20 P. = 2500 P.; $\sqrt{2500} = 50$ = no. rods on one side; $4 \times 50 = 200$ = no. rods in whole margin; $4 \times 200 = 800$ = no. hills on the margin.

PROBLEM III. — PAGE 256.

2. Ans. $8\frac{4}{5}$ feet.

4. 266 A. 3 R. 8 P. = 2668 chains; $2668 \div 46 = 58$.

PROBLEM IV. — PAGES 256, 257.

3. Ans. 80 ft. 8 in.

4. 1 A. 3 R. 34 P. $5\frac{1}{2}$ yd. = 9504 yd.; $9504 \div 96 = 99$; $99 \div \frac{11}{2} = 18$ = no. rods.

PROBLEM V. — PAGE 257.

2. Ans. 101 A. 1 R.
3. Ans. 2 A. 1 R. $4\frac{1}{2}$ P.

PROBLEM VI. — PAGES 257, 258.

2. Ans. 38.1838 rods.
3. 57.8 acres = 578 chains; $\sqrt{578 \times 2} = 34$.

PROBLEM VII. — PAGE 258.

2. Ans. 10 rods.

3. $20 \times 1\frac{1}{4} = 25$ = no. sq. ft. in the original board; $25 - 5 = 20$ = no. sq. ft. after the first piece was sawed off; $20 \div \frac{5}{4} = 16$ = length of the remainder, in feet. $20 - 8 = 12$ = no. sq. ft. in the last piece; $12 \div 16 = \frac{3}{4}$ = its width, in feet; $\frac{3}{4}$ ft. = 9 inches.

4. 4 A. = 640 rods; 5 A. 2 R. = 880 rods. $640 \div 64 = 10$ = no. rods distant from the longer side; $50 - 10 = 40$ = width of the remainder, in rods; $880 \div 40 = 22$ = no. of rods distant from the shorter side; $64 - 22 = 42$ = length of the last remainder, in rods; $42 \times 40 = 1680$ = no. sq. rods after the two portions are cut off; 1680 sq. rods = 10 A. 2 R.

Ans. 10 rods, 22 rods, and 10 A. 2 R.

PROBLEM VIII. — PAGE 259.

3. Ans. $30\frac{3}{8}$ sq. ft.
4. Ans. 8 A. 7.72 P.

PROBLEM IX. — PAGES 259 - 261.

2. $\frac{1}{2}(15 + 16 + 21) = 26$; $26 - 15 = 11$, $26 - 16 = 10$, $26 - 21 = 5$; $26 \times 11 \times 10 \times 5 = 14300$; $\sqrt{14300} = 119.5826$. Ans. 119.58 sq. ft.

3. $\frac{1}{2}(834 + 658 + 423) = 957.5$; $957.5 - 834 = 123.5$, $957.5 - 658 = 299.5$, $957.5 - 423 = 534.5$; $957.5 \times 123.5 \times 299.5 \times 534.5 = 18929985290.9375$, the square root of which is 137586.3. 137586.3 sq. links = 1.375863 A. = 1 A. 1 R. 20 P. 4 yd. 1.6 ft.

If the student is accustomed to the use of logarithms, he can employ them with advantage in solving examples similar to the one just explained.

4. The half sum is 22.5, and each remainder is 7.5. $22.5 \times 7.5 \times 7.5 \times 7.5 = 9492.1875$; $\sqrt{9492.1875} = 97.428$. The area may also be expressed thus: $\sqrt{3\,(7.5)^4} = (7.5)^2\sqrt{3} = (\frac{15}{2})^2\sqrt{3}$. Ans. 97.428 sq. yards.

5. $\frac{1}{2}(432 + 263 + 342) = 518.5$; $518.5 - 432 = 86.5$, $518.5 - 263 = 255.5$, $518.5 - 342 = 176.5$; $518.5 \times 86.5 \times 255.5 \times 176.5 = 2022555661.4375$, the square root of which is 44972.8. As the parallelogram contains two equal triangles, its area is 2×44972.8 sq. ft. $= 89945.6$ sq. ft. = 2 A. 10 P. 11.46 yd.

6. $\frac{1}{2}(25 + 40 + 40) = 52.5$, one remainder is 27.5, and each of the other two is 12.5. $52.5 \times 27.5 \times 12.5 \times 12.5 = 225585.9375$, the square root of which is 474.9589. The area may also be expressed thus: $\sqrt{5^{10} \times .0231} = \frac{5^5}{10^2}\sqrt{231} = \frac{5^3}{2^2}\sqrt{231} = \frac{125}{4}\sqrt{231}$.

Ans. 474.9589 P. = 2 A. 3 R. 34 P. 29 yd.

7. $\frac{1}{2}(57 + 83 + 127) = 133.5$; $133.5 - 57 = 76.5$, $133.5 - 83 = 50.5$, $133.5 - 127 = 6.5$; $133.5 \times 76.5 \times 50.5 \times 6.5 = 3352335.1875$, the square root of which is 1830.9383. The area of the rhomboid is 2×1830.9383 rods $= 3661.8766$ rods = 22 A. 3 R. 21 P. 26 yd. 5 ft.

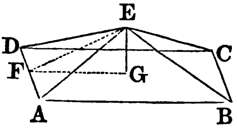
Examples 4 and 6 may also be solved by means of the principles contained in Problems X. and VIII.

PROBLEM X.—PAGE 262.

2. $53^2 - 28^2 = 2025$; $\sqrt{2025} = 45$. Ans. 45 ft.
3. $120^2 + 50^2 = 16900$; $\sqrt{16900} = 130$.
4. $25^2 + 20^2 = 1025$; $\sqrt{1025} = 32.0156$.

Ans. 32.0156 rods.

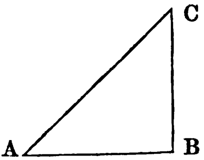
5. Let A B C D represent the garret floor, of which G is the centre. E G is the height of the roof, and F the middle of the end A D. We have the two right-angled triangles F G E and A F E, the line E F forming the hypothenuse of one and a side of the



other. $\overline{EF}^2 = \overline{FG}^2 + \overline{EG}^2 = 20^2 + 10^2 = 500$. $AE = \sqrt{\overline{EF}^2 + \overline{AF}^2} = \sqrt{500 + 12.5^2} = \sqrt{656.25} = \frac{5}{2}\sqrt{105} = 25.6174$. Ans. 25.617 feet.

6. 10 acres $= 1600$ rods; $\sqrt{1600} = 40 =$ side of the square. The diagonal of the square $= \sqrt{40^2 + 40^2} = \sqrt{3200} = 40\sqrt{2} = 56.5685$, and the distance from the centre to either corner $= \frac{56.5685}{2} = 28.2843$. $40 - 28.284 = 11.716$.

PROBLEM XI. — PAGE 263.



The following is one method of demonstrating the truth of the rule given: Let A B C be a right-angled triangle; then $\overline{AC}^2 = \overline{BC}^2 + \overline{AB}^2$ (Prop. XI. Bk. IV.). Adding $\overline{AC}^2 + 2\,AC \times BC$ to each side, $2\,\overline{AC}^2 + 2\,AC \times BC = \overline{AC}^2 + 2\,AC \times BC + \overline{BC}^2 + \overline{AB}^2$. Separating into factors, $2\,AC\,(AC + BC) = (AC + BC)^2 + \overline{AB}^2$. Dividing by $2\,(AC + BC)$, $AC = \frac{(AC + BC)^2 + \overline{AB}^2}{2\,(AC + BC)}$.

Another method of proof is used in the solution of Problem 75, Book XIII.

2. $90^2 + 50^2 = 10600$; $10600 \div (90 \times 2) = 58\frac{8}{9}$.
Ans. $58\frac{8}{9}$ miles.

3. $100^2 + 40^2 = 11600$; $11600 \div (100 \times 2) = 58$; $100 - 58 = 42$.

PROBLEM XII. — PAGE 264.

2. $17272 \times 2 = 34544$; $34544 \div 136 = 254$.
Ans. 254 yards.

3. $46.25 \times 2 = 92.5$; $92.5 \div 5.2 = 17.788$.
Ans. 17.788 chains.

4. 30 A. 3 R. 27 P. = 4947 rods; $4947 \times 2 \div 97 = 102$.

PROBLEM XIII. — PAGES 264, 265.

2. $\frac{1}{2}(2\frac{1}{4} + 1\frac{1}{2}) = \frac{15}{8}$; $\frac{15}{8} \times 16 = 30$. Ans. 30 sq. ft.

3. $\frac{1}{2}(786 + 473) = 629.5$; $629.5 \times 986 = 620687$; 620687 sq. links = 6.20687 A. = 6 A. 33 P. 3 yd.

4. $\frac{1}{2}(83 + 101) = 92$; $92 \times 60 = 5520$.
Ans. 5520 P. = 34 A. 2 R.

PROBLEM XIV. — PAGES 265, 266.

2. $25 \times 5 = 125$ = perimeter; $125 \times 8.602 = 1075.25$.
Ans. 1075.25 sq. ft. = 3 P. 28 yd. $6\frac{1}{2}$ ft.

3. $19.263 \times 7 = 134.84$ = perimeter; $134.84 \times 10 = 1348.4$; 1348.4 sq. chains = 134.84 A. = 134 A. 3 R. 14 P.

PROBLEM XV. — PAGES 266, 267.

2. $37^2 = 1369$; $1369 \times 1.7204774 = 2355.3336$.
Ans. 2355.334 sq. yards.

3. $27^2 = 729$; $729 \times 9.36564 = 6827.55$; 6827.55 sq. yards = 1 A. 1 R. 25 P. 21 yd. 2.7 ft.

4. $15.5^2 = 240.25$; $240.25 \times 4.828427 = 1160.0296$.
Ans. 1160.03 sq. ft. = 128 yd. 8.03 ft.

5. $2286 \div 9 = 254$; $254^2 = 64516$; $64516 \times 6.1818242 = 398826.57$; 398826.57 sq. ft. = 9 A. 24 P. 28 yd.

PROBLEM XVI. — PAGE 267.

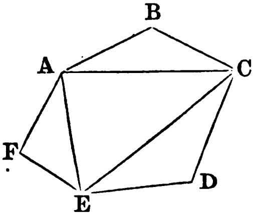
2. $1039.23 \div 2.598076 = 400$; $\sqrt{400} = 20$.
Ans. 20 feet.

3. 7 P. 18 yd. 5 ft. 128.55 in. = 298604.55 in.; $298604.55 \div 7.6942088 = 38809$; $\sqrt{38809} = 197$; 197 in. = 16 ft. 5 in.

PROBLEM XVII. — PAGES 267, 268.

2. $\frac{1}{2}(16 + 18) = 17$; $42 \times 17 = 714$. Ans. 714 sq. feet.

3. We first find the area of the triangle A B C. $\frac{1}{2}(536 + 498 + 918) = 976$; the differences between 976 and each side, respectively, are 440, 478, and 58; $976 \times 440 \times 478 \times 58 = 11905794560$, the square root of which is 109113.7 = area of the triangle A B C.



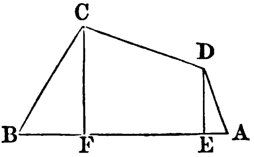
$\frac{1}{2}(918 + 1048 + 652) = 1309$; the differences between 1309 and each side, respectively, are 391, 261, and 657; 1309×391

$\times 261 \times 657 = 87765186663$, the square root of which is 296251.9 = area of the triangle A C E.

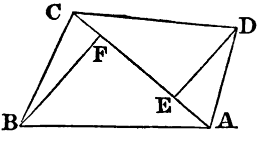
$\frac{1}{2}(620 + 580 + 1048) = 1124$; the differences between 1124 and each side, respectively, are 504, 544, and 76; $1124 \times 504 \times 544 \times 76 = 23421210624$, the square root of which is 153039.9 = area of the triangle C D E.

$\frac{1}{2}(652 + 398 + 492) = 771$; the differences between 771 and each side, respectively, are 119, 373, and 279; $771 \times 119 \times 373 \times 279 = 9548043183$, the square root of which is 97714.1 = area of the triangle A E F.

The area of the whole hexagon $= 109113.7 + 296251.9 + 153039.9 + 97714.1 = 656119.6$ sq. links $= 6.561196$ A. = 6 A. 2 R. 9 P. 23 yd. 8.4 ft.



4. $\frac{110 \times 352}{2} = 19360$ = area of the triangle A E D. $\frac{595 + 352}{2} \times (745 - 110) = 473.5 \times 635 = 300672.5$ = area of the trapezoid C D E F. $\frac{(1110 - 745) \times 595}{2} = \frac{365 \times 595}{2} = 108587.5$ = area of the triangle C F B. $19360 + 300672.5 + 108587.5 = 428620$ = area of the quadrilateral A B C D. 428620 sq. links = 4.2862 A. = 4 A. 1 R. 5 P. 24 yd.



5. Since A E D is a right-angled triangle, $ED = \sqrt{\overline{AD}^2 - \overline{AE}^2} = \sqrt{48400 - 10000} = \sqrt{38400} = 80\sqrt{6} = 195.9591$. As CFB is a right-angled triangle, $BF = \sqrt{\overline{BC}^2 - \overline{CF}^2} = \sqrt{70225 - 4900} = \sqrt{65325} = 255.5876$. $\frac{1}{2}(195.9591 + 255.5876) = 225.77335$; $225.77335 \times 378 = 85342.33$. 85342.33 sq. yards, reduced, gives the following

Ans. 17 A. 2 R. 21 P. 7.1 yd.

PROBLEM XVIII. — PAGES 268, 269.

2. 3.1416×628 links $= 1972.92$ links $= 1.97292$ fur. = 1 fur. 38 rd. 5 yd. $1\frac{1}{2}$ in., nearly.

3. 3.1416×7912 m. $= 24856.3392$ m. To obtain a more accurate result, we multiply by 3.14159265, which gives .06 of a mile less. Ans. 24856.28 miles.

4. .31831 × 928 rods = 295.39168 rods = 7 fur. 15 rd. 2 yd. $5\frac{1}{2}$ in.

5. $\frac{.31831 \times 1043}{2}$ ft. = 165.9986 ft. = 10 rd. 1 ft.

PROBLEM XIX. — PAGES 269, 270.

2. 31° 15′ = 31.25°; 31.25 × .01745 × 12 = 6.54.
Ans. 6.54 yards.

3. 12° 10′ = $12\frac{1}{6}$°; $12\frac{1}{6}$ × .01745 × 10 = 2.1231.

4. 57° 17′ $44\frac{1}{2}$″ = 57.2957°; 57.2957 × .01745 × 25 = 25.

PROBLEM XX. — PAGES 270, 271.

2. 355 × $1\frac{1}{4}^3$ = 10028.75. Ans. 10028.75 feet.

3. 281.5^2 = 79242.25; 79242.25 × 3.1416 = 248947; 248947 sq. links = 2.48947 A. = 2 A. 1 R. 38 P. 9 yd. 5 ft.

4. 39.25075^2 = 1540.62137; 1540.62137 × 3.1416 = 4840; 4840 yd. = 1 A. Ans. 1 acre.

5. 751^2 = 564001; 564001 × .7854 ÷ 2 = 221483; 221483 sq. ft. = 5 A. 13 P. 16 yd.

PROBLEM XXI. — PAGES 271, 272.

2. 1 A. = 160 P.; 160 ÷ .785398 = 203.718; $\sqrt{203.718}$ = 14.273; 14.273 2 = 7.1365. Using .7854 we obtain 7.136 rd. Ans. 7.1365 rods.

3. 6 A. 1 R. 27 P. 18.2 yd. = 31085 yd.; 31085 ÷ .0795775 = 390625; $\sqrt{390625}$ = 625. Using .07958 we obtain nearly the same result.

4. 7856 ÷ .0795775 = 98721; $\sqrt{98721}$ = 314.2.
Ans. 314.2 feet.

5. 32 × 18 = 576; 576 ÷ .7854 = 733.4; $\sqrt{733.4}$ = 27.081; 27.081 rd. = 27 rd. 1 ft. 4 in.

PROBLEM XXII. — PAGE 272.

2. 90 × $1\frac{1}{2}^2$ = 5040. Ans. 5040 sq. yds.

3. 137° 20′ = $137\frac{1}{3}$°. 456^2 = 207936; 207936 × 3.141593 = 653250.3 = area of the circle; then 360° : $137\frac{1}{3}$° :: 653250.3 : 249202.9; 249202.9 sq. links = 2.492029 A. = 2 A. 1 R. 38 P. 21.92 yd. If we use 3.1416, we obtain 2 A. 1 R. 38 P. 21.95 yd.

Problem XXIII.—Pages 272–274.

3. $200^2 = 40000$; $40000 \times .785398 = 31415.92$ = area of the circle; then 360° : 100° :: 31415.92 : 8726.64, the area of the sector. $153.208 \div 2 = 76.604$; $\sqrt{100^2 - 76.604^2} = \sqrt{4131.827} = 64.279$ = altitude of the triangle; $64.279 \times 76.604 = 4924.03$ = area of the triangle. $8726.64 - 4924.03 = 3802.61$. The area of the triangle may also be found by Prob. IX. Ans. 3802.6 sq. ft.

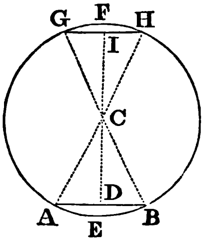
4. $4 \div 102 = .039$; taking the areas corresponding to the heights .03 and .04 in the table, we find their difference to be .00367. .01 : .001 :: .00367 : .00037 ; .00037 subtracted from the area corresponding to the height .04, gives .01017, the area corresponding to the height .039 ; $102^2 = 10404$; $10404 \times .01017 = 105.8$.

The above method gives only an approximate value to the area corresponding to the height .039, and the height itself is carried only to the place of thousandths. The area, more accurately expressed, is $10404 \times .01023 = 106.4$; hence the former result is true only to the place of units.

5. $100^2 = 10000$; $10000 \times 3.141593 = 31415.93$ = area of the circle; then 360° : 160° :: 31415.93 : 13962.63, the area of the sector. $196.9616 \div 2 = 98.4808$; $\sqrt{100^2 - 98.4808^2} = \sqrt{301.5320} = 17.3647$ = altitude of the triangle; $98.4808 \times 17.3647 = 1710.09$ = area of the triangle. $13962.63 - 1710.09 = 12252.54$ = area of the segment.

Ans. 12252.54 sq. ft.

Problem XXIV.—Pages 274, 275.



2. Let A B = 20 = chord of 60°, and G H = 14.66 = chord of 43°. Having drawn the radii A C, B C, C H, and C G, draw also the perpendiculars C D and C I. $40^2 = 1600$; $1600 \times .7854 = 1256.64$ = area of the circle. 360° : 60° :: 1256.64 209.44, area of the sector A E B C ; $20 \div 2 = 10$ = A D ; $\sqrt{20^2 - 10^2} = \sqrt{300} = 17.3205$ = C D ; $17.3205 \times 10 = 173.205$

= area of the triangle A B C; 209.44 — 173.21 = 36.23 = area of the segment A E B.

360° : 43° : : 1256.64 : 150.10, area of the sector G F H C; 14.66 ÷ 2 = 7.33 = G I; $\sqrt{20^2 - 7.33^2} = \sqrt{346.2711}$ = 18.608 = C I; 18.608 × 7.33 = 136.40 = area of the triangle C G H; 150.10 — 136.40 = 13.70 = area of the segment G F H. 36.23 + 13.70 = 49.93; 1256.64 — 49.93 = 1206.71 = area of the zone A B H G. If the two chords were both drawn on the same side of the centre C, the difference of the areas of the two segments, or 22.53, would be the area of the zone. Ans. 1206.71 sq. yd.

3. 8 ÷ 20 = .4, and the area, taken from the table, is .29337; 6 ÷ 20 = .3, and the corresponding area is .19817; .29337 + .19817 = .49154; .7854 — .49154 = .29386; 20^2 × .29386 = 117.544.

PROBLEM XXV. — PAGE 275.

2. 5.072 ÷ 24 = .2113; the area corresponding to .21 in the table is .11990, and the difference between the areas of .21 and .22 is .00821, then .01 : .0013 : : .00821 : .00107; .11990 + .00107 = .12097 = area corresponding to .2113; 24^2 × .12097 = 69.68 = area of the segment of the first circle. 8 ÷ 20 = 40, the corresponding area of which is .29337; 20^2 × .29337 = 117.35 = area of the segment of the second circle; 117.35 — 69.68 = 47.67 = area of the crescent. Ans. 47.67 sq. yds.

PROBLEM XXVI. — PAGES 275, 276.

2. $36^2 - 24^2$ = 720; 720 × 3.1416 = 2261.95.
Ans. 2261.95 sq. feet.

3. 256^2 = 65536; 2 × 29 ft. = 58 ft. = $19.\dot{3}$ yds.; 256 + $19.\dot{3}$ = $275.\dot{3}$; $275.\dot{3}^2$ = $75808.\dot{4}$; $75808.\dot{4}$ — 65536 = $10272.\dot{4}$; $10272.\dot{4}$ × .7854 = 8068; 8068 sq. yards = 1 A. 2 R. 26 P. 21.5 yd.

PROBLEM XXVII. — PAGES 276, 277.

2. 100 × .2821 = 28.21.
3. 30 × .8862 = 26.586. Ans. 26.59 feet.
4. 500 × .2821 = 141.05.

PROBLEM XXVIII. — PAGES 277, 278.

2. $300 \times .2251 = 67.53$.

3. $36 \times .7071 = 25.4556$. Ans. 25.456 inches.

4. $1000 \times .2251 = 225.1$.

PROBLEM XXIX. — PAGES 278, 279.

2. $80 \times .2757 = 22.056$.

3. $36 \times .8660 = 31.176$. Ans. 31.18 inches.

4. $251.33 \times .2757 = 69.29$. Ans. 69.29 feet.

5. $100^2 \times .7854 = 7854$ = area of the circle; $100 \times .866025 = 86.6025 = 50\sqrt{3}$ = side of the inscribed equilateral triangle; $86.6025^2 \times .4330127 = 7500 \times .4330127 = 3247.6$ = area of the triangle; $7854 - 3247.6 = 4606.4$. (.4330127 is taken from the table, p. 266 Geom.)

PROBLEM XXX. — PAGES 279, 280.

2. $75 \times 35 \times .7854 = 2061.67$. Ans. 2061.67 sq. ft.

3. $526 \times 354 \times .7854 = 146244.62$; 146244.62 sq. in. = 112 yd. 7 ft. 84.62 in. This is accurate only to the place of units, as may be shown by using .785398.

4. $436 \times 2 = 872$; $254 \times 2 = 508$; $872 \times 508 \times .7854 = 347913.35$; 347913.35 sq. ft. = 7 A. 3 R. 37 P. 27 yd. 7 ft. By using .785398 we find that the true value is nearer 6 ft. than 7 ft.

BOOK XII.

PROBLEM I. — PAGES 281, 282.

2. $2 \times 2 + 3 \times 2\frac{1}{2} + \frac{3}{4} = 12\frac{1}{4}$; $12\frac{1}{4} \times 12 = 147$.
Ans. 147 sq. ft.

3. $25^2 \times 6 = 3750$. Ans. 3750 sq. ft.

4. $(30 + 50) \times 2 = 160$ = perimeter of the floor; $160 \times 20 = 3200$ = wall surface; 3200 sq. ft. = $355\frac{5}{9}$ sq. yards.

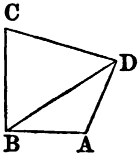
PROBLEM II.—PAGES 282, 283.

2. $\frac{1}{2}(3 + 4 + 5) = 6$; $6 - 3 = 3$, $6 - 4 = 2$, $6 - 5 = 1$; $6 \times 3 \times 2 \times 1 = 36$; $\sqrt{36} = 6$ = area of the base; $6 \times 10 = 60$.

3. $3 \times \frac{1}{2} \times 8 = 12$. Ans. 12 cu. feet.

4. $10^3 = 1000$ = solid contents of the cistern in feet; 1000 cu. ft. = 1728000 cu. in.; $1728000 \div 231 = 7480.519$.

5. Let A B C D represent the base of the prism. A B = 43, B C = 54, C D = 62, A D = 38, and B D = 70. We first find the area of the triangle A B D. $\frac{1}{2}(43 + 38 + 70) = 75.5$; $75.5 - 43 = 32.5$, $75.5 - 38 = 37.5$, $75.5 - 70 = 5.5$; $75.5 \times 32.5 \times 37.5 \times 5.5 = 506085.9375$; $\sqrt{506085.9375} = 711.397$ = the area of the triangle A B D. $\frac{1}{2}(54 + 62 + 70) = 93$; $93 - 54 = 39$, $93 - 62 = 31$, $93 - 70 = 23$; $93 \times 39 \times 31 \times 23 = 2586051$; $\sqrt{2586051} = 1608.120$ = area of the triangle B C D. $711.397 + 1608.120 = 2319.517$ = area of the base A B C D; 2319.517 sq. inches = 16.10776 sq. feet; $16.10776 \times 19 = 306.047$.



6. $4 \times 4\frac{1}{2} \times 160 = 2880$; 2880 ft. = $22\frac{1}{2}$ cords.

PROBLEM III.—PAGE 283.

2. 3×42 inches = 126 inches = 10.5 feet = perimeter of the base; $10.5 \times \frac{18}{2} = 94.5$ = convex surface; 42 inches = 3.5 feet; $3.5^2 = 12.25$; $12.25 \times .4330127$* $= 5.304$ = area of the base; $94.5 + 5.304 = 99.804$.

3. $3 \times 3 = 9$ = perimeter of the base; $9 \times \frac{20}{2} = 90$ = convex surface. Ans. 90 sq. ft.

4. 40 inches = $3\frac{1}{3}$ feet; $3\frac{1}{3} \times 2 \times 20.04 \div 2 = 66.8$ = area of the two larger triangles; 30 inches = $2\frac{1}{2}$ feet; $2\frac{1}{2} \times 2 \times 20.07 \div 2 = 50.175$ = area of the two smaller triangles; $66.8 + 50.175 = 116.975$ = convex surface; $3\frac{1}{3} \times 2\frac{1}{2} = 8.333$ = area of the base; $116.975 + 8.333 = 125.308$.

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* See Table, p. 266 of the Geometry.

PROBLEM IV. — PAGE 284.

2. $2\frac{2}{3} + 3\frac{1}{3} = 6$; $6 \times 6 \div 2 = 18$ = half the sum of the perimeters; $18 \times 16 = 288$. Ans. 288 sq. ft.

3. $\frac{18 + 34}{2} \times 5 = 130$; 130 inches $= 10\frac{5}{6}$ feet; $10\frac{5}{6} \times 11 = 119.1\dot{6}$ = convex surface; $\frac{18^2 + 34^2}{144} \times 1.7204774^{*} = 17.6827$ = area of the two bases; $119.1\dot{6} + 17.6827 = 136.849$.

PROBLEM V. — PAGES 284, 285.

2. $29^2 \times 2.5980762^{*} \div 144 = 15.1735$ = area of the base; $15.1735 \times \frac{9}{3} = 45.520$. Ans. 45.52 cu. ft.

3. $30^2 = 900$; $900 \times 25 \div 3 = 7500$.

4. $\frac{1}{2}(34 + 42 + 50) = 63$; $63 - 34 = 29$, $63 - 42 = 21$, $63 - 50 = 13$; $63 \times 29 \times 21 \times 13 = 498771$; $\sqrt{498771} = 706.2372$ = area of the base; 706.2372 sq. inches = 4.90443 sq. feet; $4.90443 \times 24 \div 3 = 39.2354$.

PROBLEM VI. — PAGE 285.

2. $(\frac{15}{12})^2 = \frac{25}{16}$ = area of one base; $(\frac{6}{12})^2 = \frac{1}{4}$ = area of other base; $\sqrt{\frac{25}{16} \times \frac{1}{4}} = \frac{5}{8}$ = mean proportional; $\frac{25}{16} + \frac{1}{4} + \frac{5}{8} = \frac{39}{16}$; $\frac{39}{16} \times \frac{20}{3} = \frac{65}{4} = 16\frac{1}{4}$. Ans. $16\frac{1}{4}$ cu. ft.

3. $(\frac{18}{12})^2 + (\frac{6}{12})^2 + (\frac{18}{12} \times \frac{6}{12}) = \frac{13}{4}$; $\frac{13}{4} \times 1.7204774^{*}$ 5.5916 = sum of the areas of the bases and mean proportional; $5.5916 \times \frac{5}{3} = 9.319$.

4. $\frac{1}{2}(21 + 15 + 12) = 24$; $24 \times 3 \times 9 \times 12 = 7776$; $\sqrt{7776} = 36\sqrt{6}$ = area of lower base; $\frac{1}{2}(14 + 10 + 8) = 16$; $16 \times 2 \times 6 \times 8 = 1536$; $\sqrt{1536} = 16\sqrt{6}$ = area of upper base; $36\sqrt{6} \times 16\sqrt{6} = 3456$; $\sqrt{3456} = 24\sqrt{6}$ = mean proportional; $36\sqrt{6} + 16\sqrt{6} + 24\sqrt{6} = 76\sqrt{6} = 186.1612$; $186.1612 \times \frac{14}{3} = 868.752$.

PROBLEM VII. — PAGES 286, 287.

2. $24 + (36 \times 2) = 96$; $96 \times \frac{22 \times 9}{6} = 3168$; 3168 cu. in. $= 1\frac{5}{6}$ cu. ft. Ans. $1\frac{5}{6}$ cu. ft.

3. $55 + (35 \times 2) = 125$; $125 \times \frac{343}{20} \times \frac{15}{6} = \frac{42875}{8} = 5359\frac{3}{8}$; $5359\frac{3}{8}$ cu. in. = 3 cu. ft. $175\frac{3}{8}$ cu. in.

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* See Table, p. 266 of the Geometry.

PROBLEM VIII. — PAGES 288, 289.

2. $24 \times 20 = 480$; $16 \times 12 = 192$; $\frac{24+16}{2} \times \frac{20+12}{2} \times 4 = 1280$; $480 + 192 + 1280 = 1952$; 1952 sq. in. $= 13\frac{5}{9}$ sq. ft.; $13\frac{5}{9} \times \frac{18}{6} = \frac{122}{3} = 40\frac{2}{3}$. Ans. $40\frac{2}{3}$ cu. ft.

3. $\frac{30}{12} \times \frac{27}{12} = \frac{45}{8}$; $\frac{24}{12} \times \frac{18}{12} = 3$; $(\frac{30}{12} + \frac{24}{12}) \times (\frac{27}{12} + \frac{18}{12}) = \frac{135}{8}$; $\frac{45}{8} + 3 + \frac{135}{8} = \frac{51}{2}$; $\frac{51}{2} \times \frac{36}{72} = \frac{51}{4} = 12\frac{3}{4}$.

Ans. $12\frac{3}{4}$ cu. ft.

4. $81\frac{1}{2} \times 55 = 4482.5$; $41 \times 29\frac{1}{2} = 1209.5$; $(81\frac{1}{2} + 41) \times (55 + 29\frac{1}{2}) = 10351.25$; $4482.5 + 1209.5 + 10351.25 = 16043.25$; $16043.25 \times 47.25 \div 6 = 126340.59375$, the capacity in cubic inches; reducing to gallons by dividing by 231, we obtain 546.929 gall.

PROBLEM IX. — PAGES 289, 290.

2. $20^2 \times 8.660254 = 3464.1016$; 3464.1016 sq. in. = 24.056 sq. ft. Ans. 24.056 sq. ft.

3. $12^2 \times 20.6457288 = 2972.985$.

PROBLEM X. — PAGES 290, 291.

2. $2^3 \times .1178513 = .9428104$. Ans. 0.9428 cu. ft.

3. $15^3 \times 2.181695 = 7363.2206$.

PROBLEM XI. — PAGES 291, 292.

2. $16 \times 21 = 336$. Ans. 336 sq. feet.

3. $4^2 \times .07958 \times 2 = 2.547$ = area of the bases; $4 \times \frac{10}{12} = \frac{10}{3} = 3.333$ = convex surface; $2.547 + 3.333 = 5.88$ = entire surface. Ans. 5.88 sq. ft.

4. $\frac{21}{12} \times 3.1416 = 5.4978$ = circumference of the base; $5.4978 \times 5\frac{1}{4} = 28.8634$ = convex surface; 1 A. = 43560 sq. feet; $43560 \div 28.8634 = 1509.18$.

PROBLEM XII. — PAGE 292.

2. $6^2 \times .0795775 = 2.86479$ = area of the base; $2.86479 \times 9 = 25.7831$.

3. $18.5^2 \times .785398 \times 8 = 2150.42$. Ans. 2150.42 cu. in.

PROBLEM XIII. — PAGES 292, 293.

2. 92 in. $= 7\frac{2}{3}$ ft.; $7\frac{2}{3} \times \frac{14}{2} = 53\frac{2}{3}$ = convex surface; $(\frac{23}{3})^2 \times .07958 = 4.677$ = area of the base; $53.667 + 4.677 = 58.344$. Ans. 58.344 sq. ft.

3. $3 \times 3.1416 \times \frac{9}{2} = 42.4116$ = convex surface; $3^2 \times .7854 = 7.0686$ = area of the base; $42.4116 + 7.0686 = 49.4802$. Ans. 49.48 sq. ft.

4. $\frac{900}{3} \times \frac{30}{6} = 1500$.

PROBLEM XIV. — PAGE 293.

2. $\frac{43 + 23}{2} \times 3.14159 \div 12 = 8.63938$ = half the sum of the circumferences; $8.63938 \times 9 = 77.7544$ = convex surface; $(43^2 + 23^2) \times .7854 \div 144 = 12.97$ = area of the bases; $77.754 + 12.97 = 90.724$. Ans. 90.724 sq. ft.

3. $19 \times 24 = 456$. Ans. 456 sq. ft.

4. $30 - 8 = 22$ = slant height of the frustum; $30 : 8 :: 10 : \frac{8}{3}$, the circumference of the base of the smaller cone, and the upper base of the frustum; $\frac{10 + \frac{8}{3}}{2} \times 22 = 139\frac{1}{3}$.

PROBLEM XV. — PAGE 294.

2. $12.5^2 \times 3.141593 = 490.874$ = area of the base; $490.874 \times \frac{63}{3} = 10308.35$. Ans. 10308.35 cu. ft.

3. $(\frac{42}{12})^2 \times .7854 = 9.62115$ = area of the base; $9.62115 \times \frac{18}{3} = 57.7269$.

By using .785398, we obtain 57.7268; hence the above answer is true only to the place of thousandths.

PROBLEM XVI. — PAGES 294, 295.

2. $38^2 + 27^2 + 38 \times 27 = 3199$; $\frac{3199}{144} \times .7854 \times \frac{11}{3} = 63.975$. Ans. 63.975 cu. ft.

3. $28^2 + 20^2 + 28 \times 20 = 1744$; $1744 \times .7854 \times \frac{40}{3} = 18263$ = solidity in cubic inches; $18263 \div 231 = 79.06$ = no. gallons.

PROBLEM XVII. — PAGE 295.

2. $30^2 \times 3.1416 = 2827.4$ = surface, in sq. inches; $2827.4 \div 144 = 19.635$. Ans. 19.635 sq. ft.

3. $7 \times 21.99 = 153.93$.

4. $7912^2 = 62599744$, which multiplied by 3.141593 gives 196662917.552192. Supposing the given length of the diameter to be exact, this result is reliable only to the place of hundreds. Taking 3.14159265 as the multiplier, we should find the surface to be nearly 22 miles less.

Ans. 196662896 sq. miles, nearly.

PROBLEM XVIII. — PAGES 295, 296.

2. $25 \times \frac{38}{12} = 79\frac{1}{6}$. Ans. $79\frac{1}{6}$ sq. ft.

3. $72 \times 3.1416 \times 24 = 5428.6848$, a result true only to the place of tenths. Multiplying by 3.141593, instead of 3.1416, we obtain 5428.6727.

4. $7912 \times 3.1415926535898 = 24856.281075202$ = circumference of a great circle. $24856.281075202 \times 327.192848 = 8132797.39568$ = surface of each frigid zone; $24856.281075202 \times 2053.468612 = 51041592.99898$ = surface of each temperate zone; $24856.281075202 \times 3150.67708 = 78314115.07768$ = surface of the torrid zone.

PROBLEM XIX. — PAGE 296.

2. 60 inches = 5 feet; $5 \times .31831 = 1.59155$ = diameter; $5 \times 1.59155 = 7.95775$ = surface of the sphere; $7.95775 \times 1.59155 \div 6 = 2.1109$. Ans. 2.1109 cu. ft.

3. $2160^3 = 10077696000$; $.523598775598 \times 10077696000 = 5276669286.45$. Ans. 5276669286.45 cu. miles.

4. $7912^3 = 495289174528$, which, multiplied by .523598775598, gives 259332805349.80493 cu. miles, as the solidity, a result reliable only as far as the units place, considering the diameter as exact.

PROBLEM XX. — PAGE 297.

2. $150° + 90° + 54° = 294°$; $2 \times 90° \times (3 - 2) = 180°$; $294° - 180° = 114°$; $114° \div 90° = 1\frac{4}{15}$; $20^2 \times 3.1416 \div 8 = 157.08$; $157.08 \times 1\frac{4}{15} = 198.97$.

Ans. 198.97 sq. ft.

PROBLEM XXI. — PAGES 297, 298.

2. $364 \times \frac{60}{6} = 3640.$ Ans. 3640 cu. ft.

3. $\frac{16}{12} \times 3 \times 3.1416 \times \frac{3}{6} = 6.2832.$ Or, $\frac{16}{12} \times (\frac{3}{2})^2 \times 2.0944 = 6.2832.$ Ans. 6.2832 cu. ft.

4. $6 \times 9^2 \times 2.0944 = 1017.88.$

PROBLEM XXII. — PAGES 298, 299.

2. $57 \times 76.5^2 \times 2.0944 = 698646.2868$ = solidity of the sector; $76.5 - 57 = 19.5$ = altitude of the cone; $76.5^2 - 19.5^2 = 5472$ = square of the radius of the base of the cone; $5472 \times 3.1416 = 17190.8352$ = area of the base of the cone; $17190.8352 \times \frac{19.5}{3} = 111740.4288$ = solidity of the cone; $698646.2868 - 111740.4288 = 586905.858$ = solidity of the segment. 586905.858 cu. in. = 339.645 cu. ft. Ans. 339.64 cu. ft.

3. $13 \times 16.75^2 \times 2.0944 = 7638.9313$ = solidity of the sector; $16.75 - 13. = 3.75$ = altitude of the cone; $16.75^2 - 3.75^2$ 266.5 = square of the radius of the base of the cone; $266.5 \times 3.1416 \times \frac{3.75}{3} = 1046.5455$ = solidity of the cone; $7638.9313 - 1046.5455 = 6592.3858.$ Ans. 6592.4 cu. ft.

In general, let d = the diameter of any sphere, and a = the altitude of its segment; then $\frac{\pi}{6} a\, d^2$ = solidity of the sector, $\sqrt{ad - a^2}$ = radius of base of the cone, $\frac{\pi}{6} a\,(\pm d^2 \mp 3ad \pm 2a^2)$ = solidity of the cone, and $\frac{\pi}{6} a\, d^2 \mp \frac{\pi}{6} a\,(\pm d^2 \mp 3ad \pm 2a^2) = \frac{\pi}{6} a^2 (3d - 2a)$ = solidity of the segment, whether less or greater than a hemisphere. The above examples might then be solved by the following rule: *From three times the diameter subtract twice the altitude of the segment, and multiply the remainder by the square of the altitude and by* .5236.

4. $(7912 \times 3) - (327.19 \times 2) = 23081.62$; $327.19^2 = 107053.2961$; $23081.62 \times 107053.2961 \times .523598775598 = 2470963500.327682 \times .523598775598 = 1293793463.319$ = solidity of each frigid zone.

We next take the frigid and temperate zones together, in order to have a spherical segment with one plane base. (Scholium, p. 298, Geom.) $327.19 + 2053.47 = 2380.66$; $(7912 \times 3) -$

$(2380.66 \times 2) = 18974.68$; $2380.66^2 = 5667542.0356$; $18974.68 \times 5667542.0356 \times .523598775598 = 107539796512.058608 \times .523598775598 = 56307705781.772$; $56307705781.772 - 1293793463.319 = 55013912318.453$ = solidity of each temperate zone.

Taking the solidity of the earth from Example 4, Problem XIX., $259332805349.805 - 2 \times 56307705781.772 = 146717393786.261$ = solidity of the torrid zone. The same answer may also be obtained by taking the three zones together, as a spherical segment, and subtracting from its solidity the sum of the solidities of the frigid and temperate zones.

PROBLEM XXIII.—PAGES 299, 300.

2. $36^2 \times 50 \times .5236 = 33929.28$. Ans. 33929 cu. ft.

3. $15^2 \times 25 \times .5236 = 2945.25$; $25^2 \times 15 \times .5236 = 4908.75$.

4. $(\frac{17}{6})^2 \times 3.5 \times .5236 = 14.712$; $3.5^2 \times \frac{17}{6} \times .5236 = 18.173$. Ans. Prolate, 14.712 cu. ft.; oblate, 18.173 cu. ft.

5. $7925.3^2 \times 7898.9 = 496132911292.901$; $496132911292.901 \times .523598775598 = 259774584886.834$, a result reliable to the place of units.

BOOK XIII.

It has not been thought expedient to present a full demonstration of all the propositions contained in the first part of this book. They involve none of those tedious operations or large numbers which require so much time and patience on the part of the teacher, when he is obliged to go through the whole work himself. Many of them can be proved in various ways, and if the different members of a class work independently of each other, they may present several distinct demonstrations of the same proposition. It will be the teacher's duty to decide whether each one is accurate in every particular; but it would obviously defeat the very object for which these exercises are introduced, were he to require that all should conform to a given model. Whatever is given, then, is, for the most part, in the form of hints and references. These may sometimes be of use by showing on what principles a demonstration or solution *may* be founded.

In referring to the Geometry, the book last named is to be understood, when no other is given.

1. This is the converse of Prop. IV. Bk. 1. It may be proved by the use of Prop. IV. Cor. 1, Ax. 2, and Prop. II.

2. Prop. VII., Prop. I., Ax. 1, Ax. 3.

3. Prop. IX.

4. Figure of Prop. XII. B O C > B D C, and B D C > B A C, by Prop. XXVII. and Ax. 8. Also by other methods.

5. Divide into triangles. Prop. V. and Cor., and Prop. XVIII.

6. Prop. IX.

7. Prop. XXXIII.

8. These points are not necessarily in the *middle* of the sides. Prove that the hypothenuses of the triangles thus formed are equal, by Ax. 3 and Prop. V. Cor.; hence the figure is equilateral. It is also right-angled, by Prop. I. Cor. 2, and Prop. XXVIII. Cor. 5, or Prop. XXVII.

9. Prop. XXIX. Cor. 1, Prop. XXXI.

10. Prop. IV., Prop. XXII., Prop. VI. and Cor., Prop. XXXI. Cor. 1, Ax. 3, Ax. 2.

11 and 12. Greenleaf's Algebra, pp. 213, 214.

13. Similar to Prop. XIII. Bk. II. Divide the equations, instead of multiplying.

14. Prop. XIV. Bk. I., Art. 152.

15. Join the centre of the circle and the extremities of any chord. Prop. IX. Bk. I., Art. 155.

16. Draw a diameter perpendicular to one of the lines. Prop. VI. Bk. III., Ax. 3, Prop. VI. Cor. 1, Prop. XXI. Cor. Bk. I.

Another method. Join A and E, first figure of Prop. XII. Bk. III. Apply Prop. XVIII. Bk. III. and Prop. XX. Bk. I.

17. Second part of Prop. XII. Bk. III.

18. Connect the two points of intersection of the circles, and connect the second point of intersection with the extremity of each diameter. Prop. XVIII. Cor. 2, Bk. III., Prop. II. Bk. I.

19. Join the vertex of the vertical angle and the intersection of the circumference with the base. Prop. XVIII. Cor. 2, Bk. III., Prop. I. Cor. 1, Bk. I., Prop. XIX. Bk. I.

20. Converse of Prop. XVIII. Cor. 4, Bk. III. It may be proved by using Prop. VII., and showing that the circumference must also pass through the fourth point.

21. Similar to the 5th.

22. The altitudes must be equal, Prop. VI. Cor. Bk. IV., and parallel, Prop. XXI. Cor. Bk. I.; hence the line joining the vertices is parallel to the base. Prop. XXXIII.

23. Prop. VII. Cor. Bk. IV. The line here joins the middle points, and must be proved parallel to the bases.

24. The steps in this vary but little from those of the 10th.

25. Draw the altitude to the middle of the base of the triangle, and complete the rectangle. The base of the triangle = two sides of the rectangle, and each of the other two sides of the triangle is greater than its altitude, or the side of the rectangle. Prop. XIV. Bk. I.

26. Figure of NOTE to Prop. XI. Bk. IV. Extend G C till it meets I M. The right-angled triangle thus formed, and B H N, are each easily proved equal to B A C, by Prop. VI. Bk. I.

27. Prop. IV. Bk. VI., Prop. XI. Cor. 4, Bk. IV. Or, divide each into equal triangles.

28. Converse of Prop. XV. Bk. IV. Figure of Prop. XVI. The line E F must disappear, and the diagonals must bisect each other; hence the quadrilateral is a parallelogram, Prop. XXXV. Bk. I.

29. Prop. VI. Cor. Bk. IV.

30. Make two of these lines bisect their respective angles, and draw the third through their intersection. Regarding it as the common base of two partial triangles, it may be proved by Prop. XIX. Bk. IV. and Prop. X. Bk. II., that it divides the third side of the original triangle into parts proportional to the other two sides; hence, Prop. XX. Bk. IV., it bisects the third angle of the triangle.

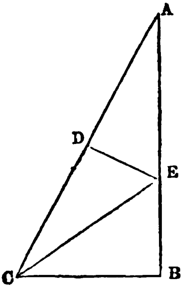
31. The three lines drawn from the centre of the inscribed circle to the vertices of the angles divide the given triangle into three others, whose bases are the sides of the triangle, and whose common altitude is the radius of the circle.

32. Each line joining the points of bisection of two sides is parallel to the third side, Prop. XVIII. Bk. IV. The triangle thus formed is therefore half of each of the three inscribed parallelograms, and equal to each of the other three triangles, or one fourth of the given triangle.

33. Draw a straight line equal to the given line, at the extremities erect two perpendiculars, each equal to the given line, and connect the extremities of the perpendiculars by a straight line. Prop. XXI. Cor. Bk. I. and Prop. XXXIII. prove the fourth side equal to the given line, and the figure an equilateral parallelogram. Prop. XXXI. proves it equiangular, or rectangular.

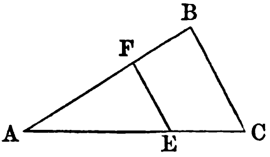
34. If the two given points are in the given line, use Prob. I. Bk. V. If either, or both of the given points are out of the given line, connect them by a straight line, and from the middle point of this straight line erect a perpendicular. Its intersection with the given line will be the point required. Prop. XV. Bk. I.

35. Draw the line A B, equal to the given sum, and making with the given base B C the angle B equal to the given angle. Connect A and C, and draw C E, making angle A C E = angle C A E. B C E will be the triangle required. Prop. VIII. Bk. I. The point E may also be determined as in the last proposition, by drawing the perpendicular D E, bisecting A C.



36. On one side of the right angle as a base, describe an equilateral triangle; then bisect that angle of the triangle whose vertex is the vertex of the right angle. Prob. XII. and VI. Bk. V., Prop. XXVIII. Cor. 6, Bk. I. Or,

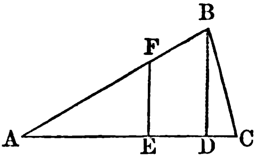
From the vertex of the right angle as a centre, describe an arc of a circle, cutting the sides of the right angle, and from these points of intersection as centres, with the same radius as before, describe arcs cutting the first arc. Lines drawn from the vertex of the right angle to these points will trisect the right angle. Prop. V. and Prop. II. Sch. 1, Bk. VI.



37. Let M and N represent the two given lines, and A C B the given triangle. By Prob. XXXIII. Bk. V., find A E, the square of which shall be to the square of A C, as N to M + N; and draw E F parallel to C B. A E F and E C B F are the parts required.

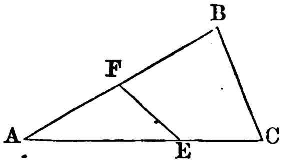
$$ACB : AEF :: \overline{AC}^2 : \overline{AE}^2 :: M + N : N.$$

By division, $ACB - AEF : AEF :: M : N.$



38. Let M and N represent the two given lines, and A C B the given triangle. Draw the altitude B D. By Prob. XXX. and Prob. XXXIII. Bk. V., find A E, the square of which shall be to the rectangle contained by A C and A D, as N to M + N; and draw E F perpendicular to A C. A E F and E C B F are the parts required.

$ADB : ACB :: AD : AC :: \overline{AD}^2 : AC \times AD$, and
$ADB : AEF :: \overline{AD}^2 : \overline{AE}^2$; hence
$ACB : AEF :: AC \times AD : \overline{AE}^2 :: M + N : N$, and
$ACB - AEF : AEF :: M : N.$



39. Let M and N represent the two given lines, A C B the given triangle, and F the given point. By Prob. XXX. and Prob. XXXIII. Bk. V., construct a square which shall be to the rectangle contained by A C and A B, as N to M + N, and (Prob. XXIX. Bk. V.) on A F construct an equivalent rectangle. Lay off A E equal to the other side of the last rectangle, and draw E F. A E F and E C B F are the required parts.

$ACB : AEF :: AC \times AB : AE \times AF :: M + N : N,$
$ACB - AEF : AEF :: M : N.$

40. Divide each side of the triangle into the same number of

equal parts, and join the points of division by lines parallel to the sides of the triangle. The number of triangles thus formed will be the square of the number of equal parts into which each side is divided. The 32d is only a particular case of this problem.

41. On the given line describe an equilateral triangle; draw straight lines bisecting the angles adjacent to the given line, and from the intersection of these bisecting lines draw two straight lines parallel to the sides of the triangle; their intersection with the base will give the points of trisection. Prop. XXII. Bk. I., Prop. VIII.

This problem may be solved by Prob. XXIII. Bk. V. Also by erecting perpendiculars, or any parallels, at the two extremities of the given line, in opposite directions, one twice as long as the other, and connecting the extremities of the perpendiculars or parallels by a straight line. The intersection of this line with the given line will give one of the points, and the other is easily found. Prop. XXII. Bk. IV.

42. Trisect the hypothenuse, draw perpendiculars from the two points of trisection to the other sides of the triangle, and join the last two points. Prop. VII. and Prop. VIII. Bk. I.

A square may also be formed by drawing lines from the middle points of the equal sides to the middle point of the hypothenuse. This square will not be equal to the former one.

43. Bisect the right angle at the centre of the circle by a straight line (Prob. VI. Bk. V.), and from this line's intersection with the circumference draw perpendiculars to the two radii. The figure is a square, because three of its angles are right angles, and the two triangles of which it is composed are equal and isosceles.

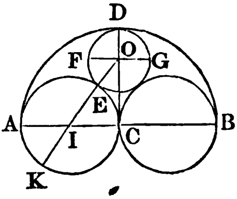
44. From any point in the given line draw a perpendicular equal to the given radius, and through the extremity of this perpendicular draw a line parallel to the given line. From the given point, with the given radius, describe an arc, and its intersection with the parallel above named will fix the centre of the required circle.

45. From the vertex of the angle opposite the perpendicular,

lay off on the hypothenuse a distance equal to the base; from this latter point draw a perpendicular to the hypothenuse, and its point of intersection with the perpendicular of the triangle will be the centre of the circle. The truth of this construction may be demonstrated by dividing the quadrilateral thus formed into two triangles, and thus proving the centre equally distant from the hypothenuse and the vertex of the right angle.

46. Construct an equilateral triangle, bisect one side, and from the vertices of the three angles as centres, with a radius equal to half the side of the triangle, describe three circles. Bisect two angles of the equilateral triangle by two straight lines, and their point of intersection will be the centre of the circle inscribed within the curvilinear space. It will also be the centre of a circle circumscribing the three equal circles, its radius being equal to the radius of the inscribed circle, plus the diameter of one of the equal circles. The demonstration is easily derived from isosceles and equal triangles, &c.

47. Join the centres O and I, and produce the line to K; also draw C D.



$OE : OC :: OC : OK$ (Prop. XXXV. Bk. IV.).

$OE : OE + OC :: OC : OC + OK$ (Prop. VII. Bk. II.).

$OE : CD :: OC : CD + KE :: OC : 2\,CD$.

$OE : OC :: CD : 2\,CD :: 1 : 2$.

$OE : CD :: 1 : 3$ (Prop. VII. Bk. II.).

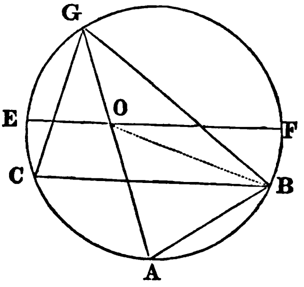
$2\,OE : CD :: 2 : 3$, or $FG : AC :: 2 : 3$.

Another demonstration. $\overline{OI}^2 = \overline{OC}^2 + \overline{IC}^2$, but $OI = IC + OE$, and $OC = 2\,IC - OE$; hence $\overline{IC}^2 + 2\,IC \times OE + \overline{OE}^2 = 4\,\overline{IC}^2 - 4\,IC \times OE + \overline{OE}^2 + \overline{IC}^2$, or $6\,IC \times OE = 4\,\overline{IC}^2$, and $3\,OE = 2\,IC$.

48. Join the centre and the point where the two lines meet on the circumference. Prop. XIV. Bk IV.

49. Having described the circumscribing circle, from any point in the circumference, as A, measure off two equal arcs, A B, A C, each subtending an angle at the centre equal to the

given angle. Join B C, and at a distance from it equal to the radius of the inscribed circle, draw E F parallel to it. From A as a centre, with a radius equal to A B, describe an arc cutting the line E F on either side of the centre, as at O, and draw A O G. Connect B and C with the point G, and B G C will be the triangle required.



The angle C G B = the given angle (Prop. XVIII. Bk. III.), and it is bisected by the line O G. Draw B O; then, since A B = A O, A O B = A B O; but A O B = O B G + A G B, and A G B = A B C ∴ A B O — A B C = O B G, or C B O = O B G, and the angle C B G is bisected by the line O B. The point O is, therefore, the centre of the inscribed circle (Prob. XVII. Bk. V.), and its distance from B C was made equal to the given radius.

50. From the vertex of one of the angles draw two diagonals, one cutting off two and the other three sides of the polygon. Prop. II. Bk. VI., Prop. III. Bk. III., Prop. XVIII. Bk. III., Prop. XX. Bk. I. By similar reasoning the diagonal which cuts off five sides may be proved parallel to that which cuts off three, and so on.

51. Prop. V. Cor. 3, Bk. VI. Each rhombus is composed of two equal triangles; thus three of these triangles form the equilateral triangle, while six of them form the hexagon.

52. Prop. IV. Bk. VI. The diameter of the circle is equal to each.

53. Make the diameter of the required circle equal to the side of the square inscribed in the given circle, that is, equal to the chord which subtends one quarter of the circumference of the given circle. Prop. IV. Bk. VI., Prop. XI. Cor. 4, Bk. IV., Prop. XIII. Cor. 2, Bk. VI.

54. Let D represent the diameter of a circle in which a regular hexagon is inscribed. Bisect the arcs subtended by the sides of the hexagon, and the chords of these semi-arcs will form

a regular dodecagon. By drawing lines from the centre of the circle to the vertices of the angles, the dodecagon will be divided into twelve equal triangles. The base of each triangle is the radius of the circumscribed circle, or $\frac{1}{2}$ D, and its altitude is one half the side of the hexagon, or $\frac{1}{4}$ D; hence the area of each triangle is $\frac{1}{16}$ D^2, and the area of the dodecagon is $\frac{12}{16}$ D^2, or $\frac{3}{4}$ D^2.

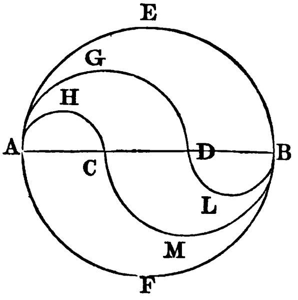
The square of the side of an equilateral triangle is to the square of the diameter of the circumscribed circle, as 3 to 4. Prop. V. Cor. 3, Bk. VI.

55. Prop. XIII. Cor. 2, Bk. VI.,* Prop. XI. Bk. IV.

56. By the last proposition, the large semicircle is equivalent to the sum of the two smaller ones. Taking away from each side of the equation the two segments of the large circle whose chords are the two shorter sides of the triangle, we have left the area of the triangle equivalent to the sum of the two crescents.

57. Prop. XIII. Cor. 1, Bk. VI.* The small semicircles may all be described on one side of the diameter, or some on one side, and some on the other.

58. Divide the diameter into any number of equal parts. Upon the first of those equal parts, as a diameter, describe a semicircle, next describe one upon the first two parts, then upon the first three parts, and so on, all on one side of the diameter. Commencing on the other side of the circle, proceed in the same way, describing all the semicircles upon the other side of the diameter. The figures thus formed, supposing the diameter to be effaced, are the required parts of the circle.



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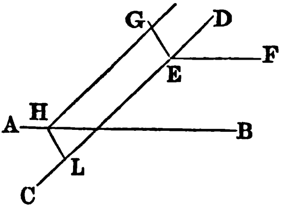
* A modified form of Prop. XI. Bk. II. may here be used to advantage. If A : C :: B : D, A : E :: B : F, &c., then A : C + E + &c. :: B : D + F + &c. The proof of this is substantially the same as that of the proposition referred to, omitting the identical equation A × B = A × B. In the 55th and 57th propositions, one of the final ratios being that of equality, the other must be also.

It is evident from the preceding proposition, that the lines A H C M B and A G D L B are each equal to the semi-circumference A E B; hence the perimeters of all the figures are equal, each being equal to the circumference of the original circle. The semicircles A H C, A G D, A E B, are to each other as $1^2, 2^2, 3^2$, or 1, 4, 9; hence A H C, A G D C H, A E B D G, are to each other as 1, 3, 5. But A F B M C, C M B L D, D L B, are to each other as 5, 3, 1; taking the sum of the corresponding terms, we find A F B M H, A H M B L G, and A G L B E to be equal in area. A similar proof would answer for any other number of parts.

59. One method is given in Prop. V. Cor., Bk. VII. Another method is as follows: Draw any straight line in the given plane, and from the given point let fall a perpendicular to that line. From the foot of the perpendicular draw another line in the plane, perpendicular to the first line. A perpendicular drawn from the given point to this last line, will be perpendicular to the plane. To prove this line perpendicular to the plane, draw through its foot a third line in the plane, parallel to the first. Prop. VI. Cor., Prop. X., and Prop. IV. Bk. VII.

60. Let A B and C D be the two lines, not in the same plane. From any point in C D draw E F, parallel to A B, and at the point E erect E G, perpendicular to the plane of C D and E F. Pass a plane through C D and E G, meeting the line A B in H, and H L, drawn from H perpendicular to C D, will also be perpendicular to A B. H L and G E are parallel (Prop. XXI. Cor., Bk. I.). Draw H G parallel to C D. The planes B H G and F E D are parallel (Prop XVI. Bk. VII.), and G E is perpendicular to B H G (Prop. XIV. Bk. VII.); hence H L is perpendicular to the plane B H G (Prop. X. Bk. VII.) and to A B (Art. 388).



61. Prop. XIII. Bk. VIII. The same three factors are used, whether a triangle or a rectangle be considered the base.

62. Prop. XIII. Bk. VIII.

63. Prop. VIII. Bk. VI., Prop. XV. Bk. VIII. Convex surface : area of base :: perim. base $\times \frac{1}{2}$ slant height : perim. base $\times \frac{1}{2}$ radius :: slant height : radius.

64. From any point of the circumference of one base, draw a line perpendicular to the plane of that base, and join the extremities of the axis and the points where this perpendicular meets the planes of the two bases. The figure thus formed will be a rectangle (Art. 388, Prop. X. Cor. 1, Bk. VII., Prop. XIII. Bk. VII.), equal to the forming rectangle (Art. 567). The moving side of the forming rectangle, which describes the convex surface of the cylinder, will, at one point of its revolution, coincide with the given perpendicular.

65. Prop. XIII. Bk. VIII., Prop. II. Bk. X.

66. Suppose the altitudes of the two solids to be divided into the same number of equal parts by planes drawn parallel to the bases, and upon the sections thus formed, as bases, construct series of inscribed and circumscribed prisms or cylinders, whose altitudes shall be the equal parts of the altitudes of the original solids. The base of any circumscribed solid will thus be the section immediately below the one which forms the base of the corresponding inscribed solid, provided the original solid is placed on its greater base, and the sections constantly decrease toward the other base. Since they have equal altitudes, any inscribed prism or cylinder of one solid, will be to the corresponding inscribed prism or cylinder of the other solid as their bases, or as the bases of the original solids. The same is true of the circumscribed series. Hence the sum of one inscribed series will be to the sum of the other inscribed series as the bases of the original solids; and the sums of the two circumscribed series will have the same ratio. Now if the altitudes be diminished by increasing the number of planes, the inscribed and circumscribed series of each solid will approach their limit, the original solid, and may be made to differ from each other by less than any given magnitude.* Hence the original solids will have the same ratio as their bases.

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* The difference between each circumscribed series and its corresponding inscribed series will be the same as the difference between the largest solid of the circumscribed series and the smallest of the corresponding inscribed

This general proposition includes several others given in the Geometry. The two solids may be two prisms, two cylinders, a prism and a cylinder, two pyramids, two cones, a pyramid and a cone, two frustums, &c.

67. By means of two right-angled triangles, it is easily proved that the square of the diameter of the sphere, or the diagonal of the cube, is equivalent to the sum of the squares of the length, breadth, and height of the cube, or three times the square of the side of the cube. Hence, the side of the cube $= \sqrt{\frac{D^2}{3}} = D\sqrt{\frac{1}{3}} = \frac{D}{3}\sqrt{3}$.

68. 14 ft. 3 in. = 171 in.; 1 sq. yard = 1296 sq. in.; $1296 \div 171 = 7\frac{11}{19}$.

69. $10^2 \times 4.828427^* = 482.8427$; 482.8427 sq. feet = 53.649 sq. yd. Ans. 53.649 yards.

70. $(16.5 \div 2) \times .31831 = 2.626$.

71. $50 \times 30 = 1500$; $25 \times 15 \times 2 = 750$; $1500 - 750 = 750$. Ans. 750 sq. ft.

72. Draw a diagonal across the trapezium, dividing it into two triangles, one of which shall be right-angled. $\sqrt{13^2 + 13.4^2} = \sqrt{348.56} = 18.67$ = length of the diagonal. $13 \times 13.4 \div 2 = 87.1$ = area of the right-angled triangle. The three sides of the other triangle are 18.67, 24, and 18. The half sum of these is 30.335, and the differences are 11.665, 6.335, and 12.335. $30.335 \times 11.665 \times 6.335 \times 12.335 = 27651.233872$; $\sqrt{27651.233872} = 166.287$; $166.287 + 87.1 = 253.387$. The true length of the diagonal is 18.66976, nearly, and had we used that in place of 18.67, the thousandths place in the answer would be found to be less than 5, instead of 7.

73. Let A B D be the equilateral triangle, with the walk surrounding it, and the circle inscribed. $\sqrt{10000 \div .433013^*} = \frac{200}{\sqrt[4]{3}} = 151.967$ = A B, the side of the triangle; $\sqrt{151.967^2 - 75.9835^2} = 100\sqrt[4]{3} = 131.607$ = altitude D E; $131.607 \div 3 = 43.869$

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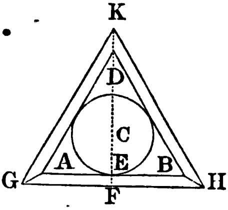
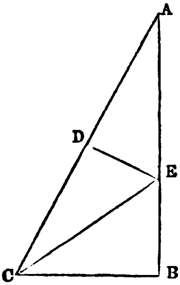
series. In the prism and cylinder this difference becomes 0, and in the pyramid and cone the smallest of the inscribed series becomes 0, and the difference becomes equal to the largest of the circumscribed series. In any of these forms the difference can be made less than any given magnitude, as in Prop. XVIII. Bk. VIII.

* Table, p. 266 of Geometry.



$=$ radius C E; $43.869^2 \times 3.1416 = \frac{10000}{9} \times \pi\sqrt{3} = 6046 =$ area of the circle, and of the walk; $10000 + 6046 = 16046 =$ area of the triangle G H K. $\sqrt{16046 \div .433013} = 192.501 =$ G H, a side of the triangle G H K; $6046 \div 3 = 2015.333 =$ area of the trapezoid G H B A; $\frac{151.967 + 192.501}{2} = 172.234$; $2015.333 \div 172.234 = 11.701 =$ altitude E F, or the width of the walk.

74. The area of the large triangle $= 16 \times 6 = 96$ rods. $96 : 24 :: 16^2 : 64$; $\sqrt{64} = 8 =$ length of the base. $96 : 24 :: 12^2 : 36$; $\sqrt{36} = 6 =$ perpendicular. $\sqrt{8^2 + 6^2} = 10 =$ hypothenuse.



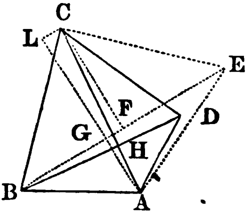
75. $\sqrt{5028\frac{4}{7} \div 3.1416} = \sqrt{1600.64} =$ radius of the pond = B C. Let A B represent the pole when standing, and C E the part broken off. Join A C and draw E D perpendicular to A C, at its middle point. As the angle E C D = C A B, and C D E = A B C, the triangles C D E and A B C are similar; hence AB : AC :: C D : C E, or $100 : \sqrt{100^2 + 1600.64} :: \frac{1}{2}\sqrt{100^2 + 1600.64} : \text{C E}$, and C E $= \frac{11600.64}{200} = 58.0032$. B E $= 100 - 58.0032 = 41.9968$.

This problem might be solved by Prob. XI. Bk. XI. It will also be seen that Prob. 35, Bk. XIII. includes such cases as this.

76. The area of the circumscribed square is double that of the inscribed; hence 800 = the area of the circumscribed square. $\sqrt{800} = 10\sqrt{8} =$ side of the circumscribed square; $\sqrt{2 \times 800} = \sqrt{1600} = 40 =$ diagonal of the circumscribed square. Or, the diagonal of the circumscribed square is just double the length of the side of the inscribed square. Ans. 40 feet.

77. Let A B C D represent the field, where A B = 25, B C = 35, C D = 31, and A D = 19. As $25^2 + 31^2 = 35^2 + 19^2$, $\overline{AB}^2 + \overline{CD}^2 = \overline{BC}^2 + \overline{AD}^2$.

If the diagonals A C, B D, of any quadrilateral are perpendicular to each other, $\overline{AH}^2 + \overline{BH}^2 = \overline{AB}^2$, $\overline{CH}^2 + \overline{DH}^2 = \overline{CD}^2$, and $\overline{AH}^2 + \overline{BH}^2 + \overline{CH}^2 + \overline{DH}^2 = \overline{AB}^2 + \overline{CD}^2$. By similar reasoning, $\overline{AH}^2 + \overline{BH}^2 + \overline{CH}^2 + \overline{DH}^2 = \overline{BC}^2 + \overline{AD}^2$; hence $\overline{AB}^2 + \overline{CD}^2 = \overline{BC}^2 + \overline{AD}^2$. The converse is also true, that is, if the sums of the squares of the opposite sides of any quadrilateral are equal, the diagonals will be perpendicular to each other; for, if AHB and CHD are acute angles, BHC and AHD must be obtuse, and we shall have $\overline{AB}^2 + \overline{CD}^2 < \overline{AH}^2 + \overline{BH}^2 + \overline{CH}^2 + \overline{DH}^2$, and $\overline{BC}^2 + \overline{AD}^2 > \overline{AH}^2 + \overline{BH}^2 + \overline{CH}^2 + \overline{DH}^2$ (Prop. XIII. Sch., Bk. IV.); whence $\overline{AB}^2 + \overline{CD}^2 < \overline{BC}^2 + \overline{AD}^2$, which is contrary to the hypothesis. If AHB and CHD are obtuse angles, $\overline{AB}^2 + \overline{CD}^2 > \overline{BC}^2 + \overline{AD}^2$, which is also contrary to the hypothesis; hence the four angles at H must be right angles.



It is evident that area $ABD = \frac{1}{2}BD \times AH$, and area $BCD = \frac{1}{2}BD \times CH$; hence area $ABCD = \frac{1}{2}BD \times AC = \frac{1}{2}\overline{AC}^2$. Draw CE perpendicular to BC, and equal to it; also draw AE. Angle DBC = angle ACE, as each + ACB = a right angle; hence triangle DBC = triangle ACE (Prop. V. Bk. I.), and AE = CD = 31. Draw the diagonal BE, also CF and AG perpendicular to it; extend AG till LG = CF, and draw CL.

$BE = \sqrt{\overline{BC}^2 + \overline{CE}^2} = 35\sqrt{2} = 49.4975$; $BF = \frac{35}{2}\sqrt{2} = 24.7487 = CF$, because BCE is both isosceles and right angled. The sides of ABE being given, its area is found by Prob. IX. Bk. XI. to be $\frac{7}{2}\sqrt{8449} = 321.7145$; hence $AG = 7\sqrt{8449} \div 35\sqrt{2} = \sqrt{168.98}$, or $2 \times 321.7145 \div 49.4975 = 12.9992$. $BG = \sqrt{\overline{AB}^2 - \overline{AG}^2} = \sqrt{625 - 168.98} = \sqrt{456.02} = \frac{151}{10}\sqrt{2} = 21.3546$; $BF - BG = \frac{35}{2}\sqrt{2} - \frac{151}{10}\sqrt{2} = \frac{12}{5}\sqrt{2}$, or $24.7487 - 21.3546 = 3.3941 = GF = LC$. $AG + CF = \sqrt{168.98} + \frac{35}{2}\sqrt{2}$, or $12.9992 + 24.7487 = 37.7479 = AL$; $\overline{AC}^2 = \overline{AL}^2 + \overline{LC}^2 = 168.98 + 7\sqrt{8449} + 612.5 + 11.52 = 793 + 7\sqrt{8449}$, or $1424.909 + 11.52 =$

1436.429, and area A B C D $= \frac{1}{2}\overline{AC}^2 = 718.2145$ P. = 4 A. 1 R. 38.2145 P.

If A B = a, B C = b, C D = c, and A D = d, the algebraic formula for the area is, area A B C D $= \frac{a^2 + c^2 \pm \sqrt{4b^2d^2 - (c^2 - a^2)^2}}{4}$ $= \frac{793 \pm 7\sqrt{8449}}{2} = 718.2145$.

78. The circumference of the outer track being twice as great as that of the inner, its radius must also be twice as great. The difference of these radii must be 5 feet; hence the radii are 5 and 10 feet, and the diameters 10 and 20 feet. 10 × 3.14159 = 31.4159; 31.4159 × 2 = 62.8318.

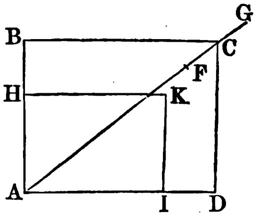
79. $(22 \div 3.1416 - 2) \times 3.1416 = 22 - 2 \times 3.1416 = 15.7168$. Ans. 15.7168 inches.

80. In every such pyramid, the height is equal to the semi-diagonal of the base, and twice the square of either is equal to the area of the base. $490^2 \times 2 = 480200$ = area of the base; 480200 ft. = 11 A. 3 P. 223¼ ft.

81. 840 × 612 × .7854 = 403758 = area enclosed. 403758 sq. links = 4 A. 6 P. 840 links = 6652.8 inches = 554.4 feet, and 612 links = 4847.04 inches = 403.92 feet; 554.4 × 403.92 = 223933.248; $(554.4 + 2\frac{1}{3}) \times (403.92 + 2\frac{1}{3}) = 226174.772\dot{4}$; $226174.772\dot{4} - 223933.248 = 2241.52\dot{4}$; $2241.52\dot{4} \times .7854 = 1760.49$. To find the area occupied by the wall, we have taken the difference between the areas of the outside and inside ellipses, and it is somewhat easier to take the difference before multiplying by .7854.

82. 1 A. = 160 P.; $\sqrt{160 \div 2} = \sqrt{80} = 8.944$ = semi-diagonal of the square. Ans. 8.944 rods.

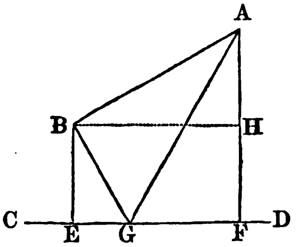
83. Let A B C D be the given rectangle. Draw the diagonal A C, and lay off C F = B C — A B, also extend A C till C G = B C — A B. Then lay off A H = ½ A F, and A I = ½ A G, and complete the rectangle A H K I, which will be half the original one, A B C D. For, A I × A H $= \frac{1}{2}$ A G × $\frac{1}{2}$ A F $= \frac{1}{2}$[A C + (B C — A B)] × $\frac{1}{2}$[A C — (B C — A B)]



$= \frac{1}{4}[\overline{AC}^2 - (BC - AB)^2] = \frac{1}{4}[\overline{AC}^2 - (\overline{AB}^2 + \overline{BC}^2) + 2AB \times BC] = \frac{1}{4}[2AB \times BC] = \frac{1}{2}[AB \times BC]$. Hence BH or ID is the width of the walk. To find the numerical value of BH, we first find AC, which is $\sqrt{80^2 + 100^2} = \sqrt{16400} = 20\sqrt{41} = 128.0625$. $BC - AB = 100 - 80 = 20$; hence $AF = 128.0625 - 20 = 108.0625$, and $AH = 108.0625 \div 2 = 54.0312$. $BH = 80 - 54.0312 = 25.9688 = ID$.

This problem may be readily solved algebraically. If $x = BH = ID$, then $(100 - x)(80 - x) = \frac{100 \times 80}{2}$, or $x^2 - 180x = -4000$, whence $x = 90 \pm 10\sqrt{41} = 25.9688$. This form shows that the width of the walk is equal to half the difference between the diagonal and the sum of the length and breadth of the rectangle.

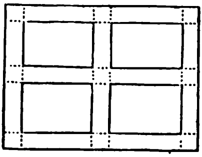
84. Let A and B represent the positions of the two trees, and CD the wall; then the perpendicular $AF = 100$, the perpendicular $BE = 50$, and the distance $AB = 100$. To find EF, draw BH parallel to it. $AH = 100 - 50 = 50$; $BH = \sqrt{100^2 - 50^2} = \sqrt{7500} = 50\sqrt{3} = 86.6025 = EF$. The person will run from one tree to G, and then to the other tree. As the angles AGF and BGE are equal, the right-angled triangles AFG and BEG are similar; that is, since $AF = 2BE$, $GF = 2EG = \frac{2}{3}EF = \frac{100}{3}\sqrt{3} = 57.735$. $AG = \sqrt{100^2 + 3333\frac{1}{3}} = \sqrt{13333\frac{1}{3}} = \frac{200}{3}\sqrt{3} = 115.4701$; $BG = \frac{1}{2}AG = \frac{100}{3}\sqrt{3} = 57.735$; $AG + BG = 100\sqrt{3} = 173.2051$.



It can readily be demonstrated that $AG + BG = 3BG = 3GF = 2EF$; hence we could obtain the above result by simply doubling EF. The triangle ABG is equal to AFG, and BG is perpendicular to AB.

85. Let the accompanying cut represent the arrangement of the walks in the park, without including the border. It will be seen that there are three walks, each 380 feet long, and three each 280 feet long, making in all 1980 feet. This length is too great, however, by nine times the width of the walk, as the in-

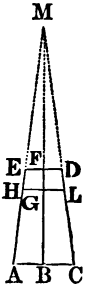
tersections form nine squares, each of which is common to two walks. The area of the whole park = 300 × 400 ft. = 120000 ft.; hence, putting w for the width of the walk, $1980\,w - 9\,w^2 = 12000$. Dividing both sides by 9, $220\,w - w^2 = \frac{12000}{9}$; changing the signs and adding 12100 to each side, $w^2 - 220\,w + 12100 = 12100 - \frac{12000}{9} = \frac{96900}{9}$; extracting the square root, $w - 110 = \sqrt{\frac{96900}{9}} = \pm \frac{10}{3}\sqrt{969} = \pm 103.7625$; $w = 110 - 103.7625 = 6.2375$.



86. The linear dimensions of the large pile are to those of each small one, as 2 to 1, hence the solid contents are to each other as 2^3 to 1, or as 8 to 1. The large pile will be to the sum of the two small ones as 8 to 2, or as 4 to 1. The same result may also be obtained by finding the actual contents of each.

Ans. One Fourth.

87. Let A C D E represent the board, and H L D E the part to be cut off. Extend A E and C D till they meet in M, and draw the altitude B M. A C = 8 in., E D = 6 in., and B F = 10 ft. A C : E D :: B M : F M, and by division A C — E D : E D :: B F : F M, or 8 — 6 in. : 6 in. :: 10 ft. : F M; hence $FM = \frac{6 \times 10}{8 - 6} = 30$ feet. Area $MED = \frac{6}{12} \times \frac{30}{2} = \frac{15}{2}$ feet, and area $MHL = \frac{15}{2} + 1 = \frac{17}{2}$ feet; hence $\frac{17}{2} : \frac{15}{2} :: \overline{GM}^2 : \overline{FM}^2$, or $\sqrt{17} : \sqrt{15} :: GM : 30$, and $GM = \frac{30\sqrt{17}}{\sqrt{15}} = 2\sqrt{255} = 31.93744$. $GF = 31.93744$ ft. — 30 ft. = 1.93744 ft. = 23.2493 in.



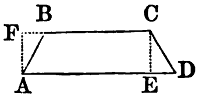
To solve this problem by algebra, put $x = GF$, and we find that area $HLDE = \frac{60 + x}{120}\,x = 1$, and $x^2 + 60\,x = 120$, which gives $x = 2\sqrt{255} - 30$, as before.

88. Conceive the timber to be extended till the whole form a pyramid, and let the last figure represent a section of it. 9 — 6 : 6 :: 10 : F M, therefore $FM = \frac{6 \times 10}{9 - 6} = 20$ feet. Solidity $MED = (\frac{6}{12})^2 \times \frac{20}{3} = \frac{5}{3}$ ft., and solidity $MHL = \frac{5}{3} + 1 = \frac{8}{3}$ ft.; hence $\frac{8}{3} : \frac{5}{3} :: \overline{GM}^3 : \overline{FM}^3$, or $2 : \sqrt[3]{5} :: GM : 20$, and $GM = \frac{40}{\sqrt[3]{5}} = 8\sqrt[3]{25} = 23.39214$. $GF = 23.39214 - 20 = 3.39214$.

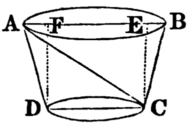
To solve this problem by algebra, let $GF = x$; then $HL = \frac{1}{2} + \frac{x}{40}$, and solidity $HLDE = \left(\frac{3}{4} + \frac{3x}{80} + \frac{x^2}{1600}\right) \times \frac{x}{3} = 1$, or $x^3 + 60x^2 + 1200x = 4800$. Adding 8000 to each side, and extracting the cube root, $x + 20 = 8\sqrt[3]{25}$, and $x = 8\sqrt[3]{25} - 20$, as before.

89. The balls may be arranged in various ways. The lines joining the centres of adjacent balls may form the edges of cubes, triangular prisms, quadrangular pyramids, or triangular pyramids. If we arrange them according to the first plan, that is, arrange them by fours, and place each one of the second layer directly above one of the first layer, it is evident that the number which will exactly fill any cubical box must be an exact cube. The cube next above 200 is 216, and its root is 6. As each ball is $2\frac{1}{2}$ inches in diameter, the length of the box will be $6 \times 2\frac{1}{2}$ inches $= 15$ inches, and there will be space for 16 more balls.

90. In order that there shall be *no* waste, the small piece cut obliquely from one end of the board must be joined to the other end, and the board can then, without any loss, be cut into six equal trapezoids. Let $ABCD$ represent one of them; draw CE perpendicular to AD, and AF perpendicular to CB produced, and the rectangle $AFCE$ will be equivalent to the trapezoid $ABCD$. It is then evident that the area of $AFCE$ is one sixth of that of the whole board, and as it has the same width as the board, its length must be $\frac{1}{6}$ as great, or 33 inches; hence $AE = 33$ inches. Each angle of a regular hexagon is $\frac{4}{3}$ of a right angle, hence ADC and DAB are each $\frac{2}{3}$ of a right angle, and the triangles CED and AFB are each half of an equilateral triangle. $CE : ED :: \sqrt{3} : 1 :: 1 : \frac{1}{3}\sqrt{3} :: 1 : .57735$; but $CE = 11$, hence $ED = 11 \times .57735 = 6.35085$. $33 + 6.35085 = 39.35085 = AD$; $33 - 6.35085 = 26.64915 = BC$.



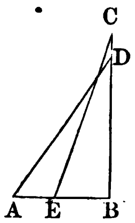
91. Let $ABCD$ represent the tub; then $AB = 48$, $BC = 30$, and $AC = 50$. Draw CE and DF perpendicular to AB. $AC^2 = AB^2 + BC^2 - 2AB \times BE$.



(Prop. XII. Bk. IV.) $50^2 = 48^2 + 30^2 - 96 \times BE$; $BE = \frac{48^2+30^2-50^2}{96} = \frac{704}{96} = \frac{22}{3} = 7\frac{1}{3}$; $AB - 2BE = 48 - 14\frac{2}{3} = 33\frac{1}{3} = FE = DC$. $CE = \sqrt{\overline{BC}^2 - \overline{BE}^2} = \sqrt{900 - 53\frac{7}{9}} = \sqrt{846\frac{2}{9}} = 29.0899$ = altitude of the frustum.

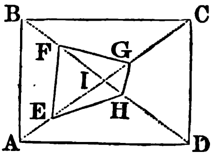
$[48^2 + 48 \times 33\frac{1}{3} + (33\frac{1}{3})^2] \times .7854 \times 29.0899 \times \frac{1}{3} = 5015\frac{1}{9} \times .7854 \times 9.69663 = 38193.7$ = capacity of the tub, in cubic inches. $38193.7 \div 231 = 165.34$ = no. gallons.

92. Let E and A be the two points where the foot of the ladder is placed, C the top of the house, and D the point 4 feet below the top. $AB = 20$, $EB = 12$. As AD and EC are equal, $\overline{AB}^2 + \overline{BD}^2 = \overline{EB}^2 + \overline{BC}^2$, or $400 + \overline{BD}^2 = 144 + \overline{BC}^2$, or $\overline{BC}^2 - \overline{BD}^2 = 256$; but $\overline{BC}^2 - \overline{BD}^2 = (BC + BD) \times (BC - BD)$, and $BC - BD = 4$; hence $BC + BD = \frac{256}{4} = 64$. The sum and difference being given, it is easy to find BC and BD. $BC = \frac{64}{2} + \frac{4}{2} = 34$. $AD = \sqrt{20^2 + 30^2} = \sqrt{1300} = 10\sqrt{13} = 36.0555$.



93. $20^3 = 8000$; $3 : 1 :: 8000 : \frac{8000}{3}$; $\sqrt[3]{\frac{8000}{3}} = \frac{20}{3}\sqrt[3]{9} = 13.8672$ = height of the upper part, or cone. $3 : 2 :: 8000 : \frac{16000}{3}$; $\sqrt[3]{\frac{16000}{3}} = \frac{20}{3}\sqrt[3]{18} = 17.4716$ = height of the cone containing the two upper shares; $17.4716 - 13.8672 = 3.6044$ = height of middle part; $20 - 17.4716 = 2.5284$ = height of lowest part.

94. Let ABCD represent the court, and EFGH the piece of water. Triangle AID : triangle EIH :: $AI \times ID : EI \times IH$. (Prop. XXVIII. Bk. IV.) A similar proportion exists between the areas and sides of the other three pairs of triangles. But the four triangles of the rectangle are all equal in area, because each has the same altitude as the one adjacent to it, regarding the semidiagonals as the bases; the products of the semidiagonals are also equal; hence the four proportions may have the same antecedents, AID and $AI \times ID$. Combining these proportions as explained in the note to propo-



sitions 55 and 57 of this Book, $AID : EIH + EIF + FIG + GIH :: AI \times ID : EI \times IH + EI \times FI + FI \times IG + IG \times IH$. Multiplying the antecedents by 4 and combining the factors of the fourth term, $4AID : EIH + EIF + FIG + GIH :: 4AI \times ID : (EI + IG) \times (FI + IH)$, or $ABCD : EFGH :: AC \times BD : EG \times FH$.

$AD = 4$ chains $= 88$ yards, and $CD = 3$ chains $= 66$ yards; $AC = \sqrt{88^2 + 66^2} = \sqrt{12100} = 110 = BD$; $EG = AC - AE - GC = 110 - 20 - 40 = 50$; $FH = BD - BF - HD = 110 - 25 - 45 = 40$; area $ABCD = 88 \times 66 = 5808$. Substituting these values in the above proportion, $5808 : EFGH :: 110^2 : 50 \times 40$, and area $EFGH = \frac{5808 \times 50 \times 40}{110^2} = 960$.

95. The surface of a sphere is expressed by πD^2 (Prop. VIII. Cor. 4, Bk. X.), and the solidity by $\frac{1}{6}\pi D^3$ (Prop. IX. Cor. 5, Bk. X.). If $\pi D^2 = \frac{1}{6}\pi D^3$, dividing both sides by πD^2 and multiplying by 6, $6 = D$.

96. $\frac{1}{4}$ acre $= 10890$ feet; $\sqrt{10890 \div .78539816} = 117.75219$ = diameter of the fortification. $(117.75219 + 26) \times 3.1415927 = 451.61083$ = length of the ditch, measuring along the middle of it. A vertical section of the ditch would be a trapezoid whose area would be $(24 + 26) \times \frac{12}{2} = 300$; hence $451.61083 \times 300 = 135483.25$ = amount of water, in cubic feet.

The same result might be obtained by considering the foundation of the fortification *with* the ditch, and the foundation *without* the ditch, as two conical frustums, the difference of whose solidities would be the solidity of the ditch, their altitude being the depth of the ditch. The diameters of the two bases of one would be 169.75219 and 167.75219, and of the other 117.75219 and 119.75219.

97. 4 acres $= 40$ chains; $\sqrt{40} = 6.3246$. $5\,A. + 4\,A. = 9\,A. = 90$ chains; $\sqrt{90} = 9.4868$. $9\,A. + 6\,A. = 15\,A. = 150$ chains; $\sqrt{150} = 12.2474$. $15\,A. + 7\,A. = 22\,A. = 220$ chains; $\sqrt{220} = 14.8324$. $\sqrt{300} = 17.3205$.

98. We find the dimensions of a cone whose slant height is 5 inches and diameter of base 4 inches. $\sqrt{5^2 - 2^2} = \sqrt{21} =$

$4.582576 =$ altitude; $4^2 \times .785398 \times 4.582576 \div 3 = 19.19549.$

Then $\sqrt[3]{19.19549} : \sqrt[3]{282} :: 4 : 9.796$, base diam.

$\sqrt[3]{19.19549} : \sqrt[3]{282} :: 5 : 12.246$, slant height.

$\sqrt[3]{19.19549} : \sqrt[3]{282} :: 4.582576 : 11.223$, the altitude.

99. The diagonal of a square whose side is unity is $\sqrt{2}$; then $\sqrt{2} - 1 : 1 :: 10 : \frac{10}{\sqrt{2}-1}$; $\frac{10}{\sqrt{2}-1} = 10(\sqrt{2} + 1) = 24.142136$ $=$ side of the square;

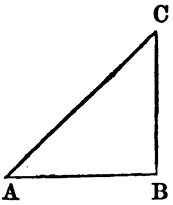
$\frac{2}{3}(24.142136)^2 = \frac{2}{3}(300 + 200\sqrt{2}) = \frac{2}{3}(582.8427) = 388.5618$ $=$ area of the octagon;

$\sqrt{388.5618 \div 4.828427^*} = 8.9707 =$ side of the octagon.

$\sqrt{5 \div .4330127^*} = 3.398 =$ side of the triangle.

BOOK XIV.

Problem VI.



Let ABC be the triangle, right-angled at B. Put $AB = a$, $BC + AC = s$, $BC = x$, and $AC = s - x$. Then

$$a^2 + x^2 = (s - x)^2 = s^2 - 2sx + x^2,$$

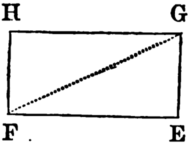
$$2sx = s^2 - a^2,$$

$$x = \frac{s^2 - a^2}{2s} = \text{altitude BC}.$$

$$s - x = \frac{s^2 + a^2}{2s} = \text{hypothenuse AC}.$$

(See Prob. XI. Bk. XI.)

Problem VII.



Let FHGE be the rectangle, and FG the diagonal. Put perim. $FHGE = a$, $FG = b$, $FE = HG = x$, and $FH = EG = \frac{a - 2x}{2}$. Then

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* Table, p. 266 of Geometry.

$$b^2 = x^2 + \left(\frac{a-2x}{2}\right)^2 = \frac{8x^2 - 4ax + a^2}{4},$$

$$8x^2 - 4ax = 4b^2 - a^2,$$

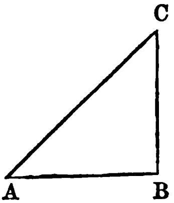
$$x^2 - \tfrac{1}{2}ax + \frac{a^2}{16} = \frac{8b^2 - a^2}{16},$$

$$x = \frac{a \pm \sqrt{8b^2 - a^2}}{4} = \text{F E}.$$

$$\frac{a-2x}{2} = \frac{a \mp \sqrt{8b^2 - a^2}}{4} = \text{F H}.$$

Problem VIII.

Let A B C be the triangle, right-angled at B. Put A B $= a$, A C — B C $= d$, B C $= x$, and A C $= x + d$. Then



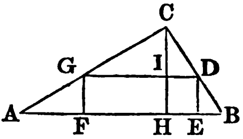
$$a^2 + x^2 = (x+d)^2 = x^2 + 2dx + d^2,$$

$$2dx = a^2 - d^2,$$

$$x = \frac{a^2 - d^2}{2d} = \text{altitude B C}.$$

$$x + d = \frac{a^2 + d^2}{2d} = \text{hypothenuse A C}.$$

Problem IX.



Let A B C represent the given triangle, and F G D E the inscribed rectangle whose area is given. If only the three sides of the triangle are given, we must first find the altitude, C H. Put A B $= b$, B C $= c$, A C $= d$, and C H $= y$. Then, by means of the two right-angled triangles, we obtain,

$$\sqrt{d^2 - y^2} + \sqrt{c^2 - y^2} = b,$$

or, $$\sqrt{d^2 - y^2} = b - \sqrt{c^2 - y^2}.$$

Squaring, $$d^2 - y^2 = b^2 - 2b\sqrt{c^2 - y^2} + c^2 - y^2,$$

or, $$2b\sqrt{c^2 - y^2} = b^2 + c^2 - d^2.$$

Squaring again,

$$4b^2c^2 - 4b^2y^2 = (b^2 + c^2 - d^2)^2.$$

Transposing, $4b^2y^2 = 4b^2c^2 - (b^2 + c^2 - d^2)^2$,

and $$y = \frac{\sqrt{4b^2c^2 - (b^2 + c^2 - d^2)^2}}{2b} = \text{C H},$$

or $$y = \frac{\sqrt{-b^4 + 2b^2c^2 + 2b^2d^2 - c^4 + 2c^2d^2 - d^4}}{2b},$$

or $$y = \frac{\sqrt{(b + c + d)(-b + c + d)(b - c + d)(b + c - d)}}{2b}.$$

The same result might be obtained by finding the area of the triangle (Art. 622, Geom.) and dividing it by one half the base.

Put h = altitude C H, a = area of the rectangle, x = F E = G D, $\frac{a}{x}$ = G F = D E. Then, the triangles A B C, G D C being similar, we have

$$b : x :: h : h - \frac{a}{x},$$

or $$bh - \frac{ab}{x} = hx,$$

and $$hx^2 - bhx = -ab.$$

Completing the square,

$$x^2 - bx + \frac{b^2}{4} = \frac{b^2}{4} - \frac{ab}{h}.$$

Evolving, $$x - \frac{b}{2} = \pm\sqrt{\frac{b^2}{4} - \frac{ab}{h}},$$

and $$x = \frac{b}{2} \pm \sqrt{\frac{b^2}{4} - \frac{ab}{h}} = \text{base F E}.$$

$$\frac{a}{x} = \frac{h}{2} \mp \sqrt{\frac{h^2}{4} - \frac{ah}{b}} = \text{altitude F G}.$$

The value of the altitude may be obtained by substituting z for $\frac{a}{x}$ and $\frac{a}{z}$ for x in the above proportion, or, as follows:

$$x = \frac{b}{2}\left(1 \pm \sqrt{1 - \frac{4a}{bh}}\right), \text{ and } \frac{a}{x} = \frac{2a}{b\left(1 \pm \sqrt{1 - \frac{4a}{bh}}\right)}.$$

Multiplying both numerator and denominator by

$$b\left(1 \mp \sqrt{1 - \frac{4a}{bh}}\right),$$

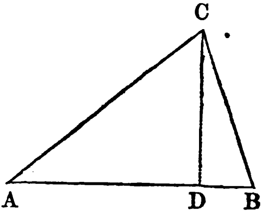
we have

$$\frac{a}{x} = \frac{h}{2}\left(1 \mp \sqrt{1 - \frac{4a}{bh}}\right) = \frac{h}{2} \mp \sqrt{\frac{h^2}{4} - \frac{ah}{b}}.$$

As bh = twice the area of the triangle A B C, it is evident that the above quantities will be imaginary when the area of the rectangle is more than half as great as that of the triangle. When $4\,a = b\,h$, the radical vanishes, and the values are $\frac{1}{2}\,b$ and $\frac{1}{2}\,h$. Hence the area of the greatest rectangle that can be inscribed in any triangle is half the area of the triangle, and its base and altitude are respectively half the base and altitude of the triangle. We also infer that when the area of the rectangle is less than half as great as that of the triangle, there will be two rectangles which will fulfil the conditions of the problem, using the same side of the triangle as base.

NOTE. — When the given triangle has an obtuse angle, it will be necessary to use the longest side as the base, otherwise a part of the rectangle will fall without the triangle. In other triangles, either side may be used as the base.

PROBLEM X.



Let A B C be the triangle, and C D the perpendicular. Put $A\,D = a$, $B\,D = b$, $\frac{A\,C}{B\,C} = c$, $B\,C = x$, and $A\,C = c\,x$.

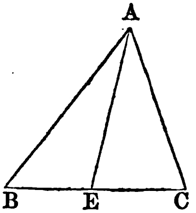
$$C\,D = \sqrt{c^2 x^2 - a^2} = \sqrt{x^2 - b^2}.$$

Squaring, $c^2 x^2 - a^2 = x^2 - b^2$,

or $c^2 x^2 - x^2 = a^2 - b^2$.

$$x = \left(\frac{a^2 - b^2}{c^2 - 1}\right)^{\frac{1}{2}} = B\,C, \text{ and } c\,x = c\left(\frac{a^2 - b^2}{c^2 - 1}\right)^{\frac{1}{2}} = A\,C.$$

PROBLEM XI.



In the triangle A B C, let $B\,C = a$, $A\,B + A\,C = b$, and $A\,E = c$. Put $A\,B = x$, and $A\,C = b - x$; then (Prop. XIV. Bk. IV.)

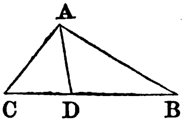
$$x^2 + (b - x)^2 = 2\,c^2 + \frac{a^2}{2},$$

or $$x^2 - b\,x = \frac{a^2 + 4\,c^2 - 2\,b^2}{4}.$$

$$x = \frac{b \pm \sqrt{a^2 - b^2 + 4\,c^2}}{2} = A\,B.$$

$$b - x = \frac{b \mp \sqrt{a^2 - b^2 + 4\,c^2}}{2} = A\,C.$$

PROBLEM XII.



Let the triangle A B C have its angle A bisected by the line A D. Put A B $= a$, A C $= b$, A D $= c$, B D $= x$, C D $= y$, and B C $= x + y$. Then (Prop. XIX. Bk. IV.),

$$x : y :: a : b, \text{ or } b\,x = a\,y, \text{ and } y = \frac{b\,x}{a}.$$

$a\,b = c^2 + x\,y$ (Prop. XXXVI. Bk. IV.) Substituting the value of y from the first equation, $a\,b = c^2 + \frac{b\,x^2}{a}$,

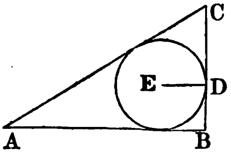
or, $$b\,x^2 = a^2\,b - a\,c^2,$$

and $$x = \left(\frac{a^2\,b - a\,c^2}{b}\right)^{\frac{1}{2}} = \text{B D}.$$

$$y = \frac{b\,x}{a} = \left(\frac{a\,b^2 - b\,c^2}{a}\right)^{\frac{1}{2}} = \text{C D}.$$

$$x + y = \left(\frac{a^2\,b - a\,c^2}{b}\right)^{\frac{1}{2}} + \left(\frac{a\,b^2 - b\,c^2}{a}\right)^{\frac{1}{2}} = (a + b)\left(\frac{a\,b - c^2}{a\,b}\right)^{\frac{1}{2}} =$$ [base B C.

PROBLEM XIII.



Let the perimeter of the triangle A B C, that is, A B + B C + A C, be represented by a, the radius of the inscribed circle E D $= c$, A B $= x$, B C $= y$, and A C $= a - x - y$. Then (Prop. 31, Bk. XIII.),

$$\frac{a\,c}{2} = \frac{x\,y}{2} \therefore y = \frac{a\,c}{x}.$$

Also, $$\sqrt{x^2 + y^2} = a - x - y = \text{A C}.$$

Squaring and cancelling, $$0 = a^2 - 2\,a\,x - 2\,a\,y + 2\,x\,y.$$

Substituting value of y, $$0 = a^2 - 2\,a\,x - \frac{2\,a^2\,c}{x} + 2\,a\,c.$$

Transposing and reducing,

$$x^2 - \left(c + \frac{a}{2}\right) x = -a\,c.$$

Completing the square,

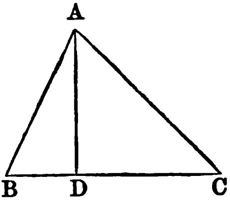
$$x^2 - \left(c + \frac{a}{2}\right) x + \left(\frac{2\,c + a}{4}\right)^2 = \frac{4\,c^2 - 12\,a\,c + a^2}{16}.$$

Evolving and transposing, $x = \frac{a + 2c \pm \sqrt{a^2 - 12ac + 4c^2}}{4} = \text{AB}.$

$$y = \frac{ac}{x} = \frac{a + 2c \mp \sqrt{a^2 - 12ac + 4c^2}}{4} = \text{BC}.$$

$$a - x - y = \frac{a - 2c}{2} = \text{AC}.$$

PROBLEM XIV.



Let $\text{AD} = a$, $\text{BC} = b$, $\frac{\text{AC}}{\text{AB}} = c$, $\text{AB} = x$, and $\text{AC} = cx$. Then $\text{BD} = \sqrt{x^2 - a^2}$, and $\text{DC} = \sqrt{c^2x^2 - a^2}$; hence, $\sqrt{x^2 - a^2} = b - \sqrt{c^2x^2 - a^2}$.

Squaring,

$$x^2 - a^2 = b^2 - 2b\sqrt{c^2x^2 - a^2} + c^2x^2 - a^2.$$

Cancelling and transposing,

$$2b\sqrt{c^2x^2 - a^2} = b^2 + c^2x^2 - x^2 = b^2 + (c^2 - 1)x^2.$$

Squaring again,

$$4b^2c^2x^2 - 4a^2b^2 = b^4 + 2b^2(c^2 - 1)x^2 + (c^2 - 1)^2x^4.$$

Transposing and reducing,

$$x^4 - \frac{2b^2(c^2 + 1)}{(c^2 - 1)^2}x^2 = \frac{-b^4 - 4a^2b^2}{(c^2 - 1)^2}.$$

Completing the square,

$$x^4 - \frac{2b^2(c^2 + 1)}{(c^2 - 1)^2}x^2 + \frac{b^4(c^2 + 1)^2}{(c^2 - 1)^4} = \frac{4b^2[b^2c^2 - a^2(c^2 - 1)^2]}{(c^3 - 1)^4}.$$

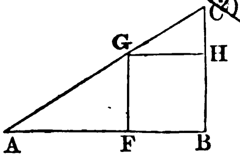
Evolving and transposing,

$$x^2 = \frac{b^2(c^2 + 1) \pm 2b\sqrt{b^2c^2 - a^2(c^2 - 1)^2}}{(c^2 - 1)^2}.$$

$$x = \frac{[b^2(c^2 + 1) \pm 2b\sqrt{b^2c^2 - a^2(c^2 - 1)^2}]^{\frac{1}{2}}}{c^2 - 1} = \text{AB}.$$

$$cx = \frac{c[b^2(c^2 + 1) \pm 2b\sqrt{b^2c^2 - a^2(c^2 - 1)^2}]^{\frac{1}{2}}}{c^2 - 1} = \text{AC}.$$

PROBLEM XV.



Let $AC = a$, $BF = b$, $AF = x$, $CH = y$, $AB = x + b$, and $BC = y + b$. As the triangles AFG, GHC are similar, we have,

$x : b :: b : y$, or $xy = b^2$, and $y = \frac{b^2}{x}$.

Also, $$(x + b)^2 + (y + b)^2 = a^2,$$

or, $$x^2 + 2bx + y^2 + 2by + 2b^2 = a^2.$$

Adding $2xy = 2b^2$,

$$x^2 + 2bx + 2xy + y^2 + 2by + 2b^2 = a^2 + 2b^2,$$

or, $$x^2 + 2xy + 2bx + y^2 + 2by + b^2 = a^2 + b^2.$$

Evolving, $$x + y + b = \pm\sqrt{a^2 + b^2}.$$

Substituting value of y, $x + \frac{b^2}{x} + b = \pm\sqrt{a^2 + b^2}$,

or, $$x^2 + (b \mp \sqrt{a^2 + b^2})\, x = -b^2.$$

Completing the square,

$$x^2 + (b \mp \sqrt{a^2 + b^2})\, x + \tfrac{1}{4}(b \mp \sqrt{a^2 + b^2})^2 = \frac{a^2 - 2b^2 \mp 2b\sqrt{a^2 + b^2}}{4}.$$

Evolving and transposing,

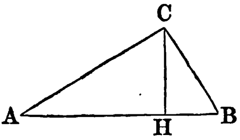
$$x = \frac{-b \pm \sqrt{a^2 + b^2} \pm (a^2 - 2b^2 \mp 2b\sqrt{a^2 + b^2})^{\frac{1}{2}}}{2}.$$

$$y = \frac{b^2}{x} = \frac{-b \pm \sqrt{a^2 + b^2} \mp (a^2 - 2b^2 \mp 2b\sqrt{a^2 + b^2})^{\frac{1}{2}}}{2}.$$

$$x + b = \frac{b \pm \sqrt{a^2 + b^2} \pm (a^2 - 2b^2 \mp 2b\sqrt{a^2 + b^2})^{\frac{1}{2}}}{2} = AB.$$

$$y + b = \frac{b \pm \sqrt{a^2 + b^2} \mp (a^2 - 2b^2 \mp 2b\sqrt{a^2 + b^2})^{\frac{1}{2}}}{2} = BC.$$

PROBLEM XVI.



Let the triangle ABC be right-angled at C. Put $AB + BC + AC = a$, $CH = b$, $AC = x$, $BC = y$, and $AB = a - x - y$. As the triangles ABC and ACH are similar,

$a - x - y : x :: y : b$, or, $xy = b(a - x - y)$, and $y = \frac{b(a - x)}{b + x}$.

Also, $$a - x - y = \sqrt{x^2 + y^2} = AB.$$

Squaring and cancelling, $a^2 - 2ax - 2ay + 2xy = 0$.
Substituting value of y,

$$a^2 - 2ax - \frac{2ab(a-x)}{b+x} + \frac{2bx(a-x)}{b+x} = 0.$$

Clearing of fractions and uniting,

$$a^2x + 2abx - 2ax^2 - 2bx^2 - a^2b = 0.$$

Transposing, &c., $x^2 - \frac{(a^2+2ab)x}{2a+2b} = \frac{-a^2b}{2a+2b}$.

Completing the square,

$$x^2 - \frac{(a^2+2ab)x}{2a+2b} + \left(\frac{a^2+2ab}{4a+4b}\right)^2 = \frac{a^4 - 4a^3b - 4a^2b^2}{(4a+4b)^2}.$$

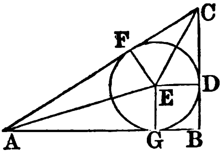
Evolving and transposing,

$$x = \frac{a}{4}\left(\frac{a+2b \pm \sqrt{a^2-4ab-4b^2}}{a+b}\right) = AC.$$

$$y = \frac{a}{4}\left(\frac{a+2b \mp \sqrt{a^2-4ab-4b^2}}{a+b}\right) = BC.$$

$$a - x - y = \frac{a}{2}\left(\frac{a}{a+b}\right) = \frac{a^2}{2a+2b} = AB.$$

PROBLEM XVII.



Let A B C be the triangle, right-angled at B. From E, the centre of the inscribed circle, draw E A and E C to the two acute angles, and E D, E F, E G, to the points of tangency. Put A C $= a$, A E — E C $= d$, A B $= x$, and B C $= \sqrt{a^2 - x^2}$. As it is necessary to find the radius of the inscribed circle before the sides can be determined, put E G $= r$, A E $= y$, E C $= y - d$, A G $= z$, and C D $=$ C F $=$ C A — A G $= a - z$. By means of the right-angled triangles A G E and E D C, the first two of the following equations are readily obtained. A C $+$ A B $+$ B C $= a + (z + r) + (a - z + r) = 2a + 2r$; hence $2ar + 2r^2 =$ twice the area of the triangle A B C. (Prop. 31, Bk. XIII.) Twice the area of the triangle A B C is also represented by A B $\times$ B C $= (z + r) \times (a - z + r) = az + ar - z^2 + r^2$. From these two expressions the third equation is derived.

$$z^2 + r^2 = y^2 \therefore y = \sqrt{z^2 + r^2}. \qquad (1)$$

$$(a - z)^2 + r^2 = (y - d)^2. \qquad (2)$$

$$a r + r^2 = a z - z^2 = \text{area, ABC}. \qquad (3)$$

Substituting value of y, (2) becomes,

$$2 d \sqrt{z^2 + r^2} = 2 a z - (a^2 - d^2).$$

Squaring, transposing, &c.,

$$4 d^2 r^2 - 4 (a^2 - d^2) z^2 + 4 (a^2 - d^2) a z = (a^2 - d^2)^2,$$

or, $$4 d^2 r^2 + 4 (a^2 - d^2)(a z - z^2) = (a^2 - d^2)^2. \qquad (4)$$

Substituting (3), $4 d^2 r^2 + 4 (a^2 - d^2)(a r + r^2) = (a^2 - d^2)^2.$

Uniting terms, $4 (a^2 - d^2) a r + 4 a^2 r^2 = (a^2 - d^2)^2.$

Completing the square,

$$4 a^2 r^2 + 4 (a^2 - d^2) a r + (a^2 - d^2)^2 = 2 (a^2 - d^2)^2.$$

Evolving, $2 a r + (a^2 - d^2) = (a^2 - d^2) \sqrt{2}.$

Transposing and dividing, $r = \dfrac{(\sqrt{2} - 1)(a^2 - d^2)}{2 a} = \text{GB}.$

Substituting value of r in (4),

$$4 (a z - z^2) = a^2 - d^2 - \frac{(3 - 2\sqrt{2})(a^2 - d^2) d^2}{a^2},$$

or, $$4 z^2 - 4 a z + a^2 = \frac{a^2 d^2 + (3 - 2\sqrt{2})(a^2 - d^2) d^2}{a^2}.$$

Evolving, $$2 z - a = \frac{\pm d \sqrt{a^2 + (3 - 2\sqrt{2})(a^2 - d^2)}}{a},$$

and, $$z = \frac{a^2 \pm d \sqrt{a^2 + (3 - 2\sqrt{2})(a^2 - d^2)}}{2 a}.$$

$x = z + r =$

$$\frac{a^2 + (\sqrt{2} - 1)(a^2 - d^2) \pm d \sqrt{a^2 + (3 - 2\sqrt{2})(a^2 - d^2)}}{2 a} = \text{AB}.$$

$\sqrt{a^2 - x^2} = a - z + r =$

$$\frac{a^2 + (\sqrt{2} - 1)(a^2 - d^2) \mp d \sqrt{a^2 + (3 - 2\sqrt{2})(a^2 - d^2)}}{2 a} = \text{BC}.$$

This problem may also be solved as follows: — Draw from C a perpendicular to AE produced, thus forming a new right-angled triangle with the hypothenuse A C, and also an isosceles right-angled triangle with the hypothenuse E C. From these triangles, A E $= \frac{d \pm \sqrt{a^2(4 - 2\sqrt{2}) - d^2(3 - 2\sqrt{2})}}{2}$, and E C $= \frac{-d \pm \sqrt{a^2(4 - 2\sqrt{2}) - d^2(3 - 2\sqrt{2})}}{2}$, may be determined, thus making all the sides of the triangle A E C known. Its altitude E F may be found as in Problem IX., and can be reduced to the form $\frac{(\sqrt{2} - 1)(a^2 - d^2)}{2a}$. Using b for this expression, (see Problem XXIII.), $x = \frac{a + 2b \pm \sqrt{a^2 - 4ab - 4b^2}}{2}$, and $\sqrt{a^2 - x^2} = \frac{a + 2b \mp \sqrt{a^2 - 4ab - 4b^2}}{2}$. To reduce these to the same form as the preceding ones, substitute for b its value, and

$$x = \frac{a^2\sqrt{2} - d^2(\sqrt{2} - 1) \pm d\sqrt{a^2(4 - 2\sqrt{2}) - d^2(3 - 2\sqrt{2})}}{2a},$$

$$\sqrt{a^2 - x^2} = \frac{a^2\sqrt{2} - d^2(\sqrt{2} - 1) \mp d\sqrt{a^2(4 - 2\sqrt{2}) - d^2(3 - 2\sqrt{2})}}{2a},$$

which expressions are identical with those previously obtained, though varying a little in form. They may also be expressed as follows: —

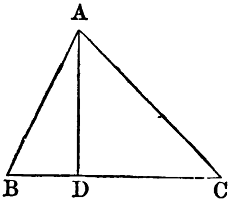
$$x = \frac{d^2 + (a^2 - d^2)\sqrt{2} \pm d\sqrt{d^2 + 2(a^2 - d^2)(2 - \sqrt{2})}}{2a},$$

$$\sqrt{a^2 - x^2} = \frac{d^2 + (a^2 - d^2)\sqrt{2} \mp d\sqrt{d^2 + 2(a^2 - d^2)(2 - \sqrt{2})}}{2a}.$$

From the formulas expressing the values of the lines A E and E C, and the sides A B and B C, it will be seen that the difference of the lines A E and E C bears the same ratio to the difference of the sides A B and B C, that the hypothenuse A C bears to the sum of the lines A E and E C, or,

$$d : \mathrm{AB} - \mathrm{BC} :: a ; \mathrm{AE} + \mathrm{EC}.$$

PROBLEM XVIII.



Let the altitude A D $= a$, the base B C $= b$, A C — A B $= d$, A B $= x$, and A C $= x + d$. Then,

$$\sqrt{x^2 - a^2} + \sqrt{(x + d)^2 - a^2} = b,$$

or, $$\sqrt{(x + d)^2 - a^2} = b - \sqrt{x^2 - a^2}.$$

Squaring,

$$x^2 + 2\,d\,x + d^2 - a^2 = b^2 - 2\,b\sqrt{x^2 - a^2} + x^2 - a^2.$$

Transposing and cancelling,

$$2\,b\sqrt{x^2 - a^2} = b^2 - d^2 - 2\,d\,x.$$

Squaring,

$$4\,b^2\,x^2 - 4\,a^2\,b^2 = b^4 - 2\,b^2\,d^2 - 4\,b^2\,d\,x + d^4 + 4\,d^3\,x + 4\,d^2\,x^2.$$

Transposing and dividing,

$$x^2 + d\,x = \frac{4\,a^2\,b^2 + b^4 - 2\,b^2\,d^2 + d^4}{4\,b^2 - 4\,d^2}.$$

Completing the square,

$$x^2 + d\,x + \frac{d^2}{4} = \frac{4\,a^2\,b^2 + b^4 - b^2\,d^2}{4\,b^2 - 4\,d^2}.$$

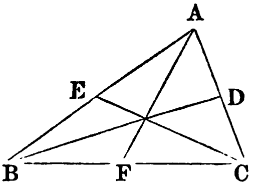
Evolving and transposing,

$$x = -\frac{d}{2} \pm \left(\frac{4\,a^2\,b^2 + b^4 - b^2\,d^2}{4\,b^2 - 4\,d^2}\right)^{\frac{1}{2}},$$

or, $$x = -\frac{d}{2} \pm \frac{b}{2}\left(\frac{4\,a^2 + b^2 - d^2}{b^2 - d^2}\right)^{\frac{1}{2}} = \text{A B}.$$

$$x + d = \frac{d}{2} \pm \frac{b}{2}\left(\frac{4\,a^2 + b^2 - d^2}{b^2 - d^2}\right)^{\frac{1}{2}} = \text{A C}.$$

PROBLEM XIX.



Let A B C be the triangle, and A F, B D, and C E the lines drawn to the middle of the sides. Put A F $= a$, B D $= b$, C E $= c$, B C $= x$, A C $= y$, and A B $= z$. Then (Prop. XIV. Bk. IV.),

(1) $y^2 + z^2 = 2a^2 + \frac{x^2}{2}$, or $-x^2 + 2y^2 + 2z^2 = 4a^2$.

(2) $x^2 + z^2 = 2b^2 + \frac{y^2}{2}$, or $2x^2 - y^2 + 2z^2 = 4b^2$.

(3) $x^2 + y^2 = 2c^2 + \frac{z^2}{2}$, or $4x^2 + 4y^2 - 2z^2 = 8c^2$.

(2) — (1) $3x^2 - 3y^2 = 4b^2 - 4a^2$.

(2) + (3) $6x^2 + 3y^2 = 4b^2 + 8c^2$.

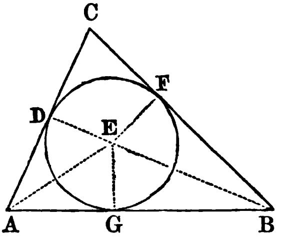
Adding these equations, $9x^2 = 8b^2 - 4a^2 + 8c^2$.

Dividing and evolving, $x = \frac{2}{3}\sqrt{2b^2 - a^2 + 2c^2}$.

In like manner, $y = \frac{2}{3}\sqrt{2a^2 - b^2 + 2c^2}$;

$z = \frac{2}{3}\sqrt{2a^2 - c^2 + 2b^2}$.

PROBLEM XX.



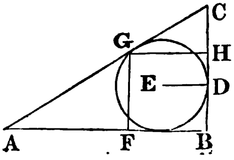
Let A B C represent the triangle, and D G F the inscribed circle. Put B C = a, A B = b, A C = c, and E F = x. As in Problem IX., we find the area of the triangle to be $\frac{1}{4}\sqrt{-a^4 + 2a^2b^2 + 2a^2c^2 - b^4 + 2b^2c^2 - c^4}$, and (Prop. 31, Bk. XIII.) it is also expressed by $\frac{x}{2}(a + b + c)$; therefore,

$$\frac{x}{2}(a+b+c) = \frac{1}{4}\sqrt{-a^4 + 2a^2b^2 + 2a^2c^2 - b^4 + 2b^2c^2 - c^4},$$

$$\text{and } x = \frac{\sqrt{-a^4 + 2a^2b^2 + 2a^2c^2 - b^4 + 2b^2c^2 - c^4}}{2(a+b+c)} = \text{E F}.$$

$$\text{Also, } x = \tfrac{1}{2}\left(\frac{(-a+b+c)(a-b+c)(a+b-c)}{a+b+c}\right)^{\frac{1}{2}}.$$

PROBLEM XXI.



Let A B C represent the right-angled triangle, B H G F the inscribed square, and E D the radius of the inscribed circle. Put E D = a, B F = b, A B = x, B C = y, and A C = $\sqrt{x^2 + y^2}$.

From the different methods of finding the area of the triangle,

$$a\,(x + y + \sqrt{x^2 + y^2}) = x\,y,$$

or, $$a\sqrt{x^2 + y^2} = x\,y - a\,x - a\,y.$$

Squaring,

$$a^2x^2 + a^2y^2 = x^2y^2 - 2\,a\,x^2\,y - 2\,a\,x\,y^2 + a^2x^2 + 2\,a^2\,x\,y + a^2\,y^2.$$

Cancelling and dividing by $x\,y$,

$$0 = x\,y - 2\,a\,x - 2\,a\,y + 2\,a^2.$$

As the triangles A B C and G H C are similar,

$$x : b :: y : y - b, \text{ or } x\,y - b\,x = b\,y, \text{ and } y = \frac{b\,x}{x - b}.$$

Substituting this value of y in the above equation,

$$0 = \frac{b\,x^2}{x - b} - 2\,a\,x - \frac{2\,a\,b\,x}{x - b} + 2\,a^2.$$

Clearing of fractions, $0 = b\,x^2 - 2\,a\,x^2 + 2\,a^2\,x - 2\,a^2\,b.$

Transposing and dividing, $x^2 - \frac{2\,a^2}{2\,a - b}\,x = \frac{-2\,a^2\,b}{2\,a - b}.$

Completing the square,

$$x^2 - \frac{2\,a^2}{2\,a - b}\,x + \frac{a^4}{(2\,a - b)^2} = \frac{a^4 - 4\,a^3\,b + 2\,a^2\,b^2}{(2\,a - b)^2}.$$

Evolving, &c.,

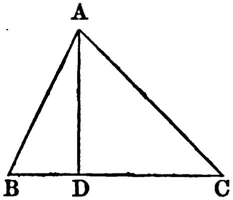
$$x = \frac{a^2 \pm a\sqrt{a^2 - 4\,a\,b + 2\,b^2}}{2\,a - b} = \text{A B}.$$

$$y = \frac{a^2 \mp a\sqrt{a^2 - 4\,a\,b + 2\,b^2}}{2\,a - b} = \text{B C}.$$

$$\sqrt{x^2 + y^2} = \frac{2\,a\,b - 2\,a^2}{2\,a - b} = \frac{2\,a\,(b - a)}{2\,a - b} = \text{A C}.$$

Some parts of the above solution may be simplified by assuming the principle proved in solving Problem XXIII., that the hypothenuse added to the diameter of the inscribed circle is equal to the sum of the other two sides. Putting $\text{A C} = x + y - 2\,a$, we get immediately, $2\,a\,x + 2\,a\,y - 2\,a^2 = x\,y.$

PROBLEM XXII.*



Let A B C represent the triangle. Put A D $= a$, B C $= b$, A B $\times$ A C $= c$, A C $= x$, and A B $= \frac{c}{x}$.

As B D = B C — D C,

$$\sqrt{\frac{c^2}{x^2} - a^2} = b - \sqrt{x^2 - a^2}.$$

Squaring,

$$\frac{c^2}{x^2} - a^2 = b^2 - 2b\sqrt{x^2 - a^2} + x^2 - a^2.$$

Transposing, &c.,

$$2b\sqrt{x^2 - a^2} = x^2 + b^2 - \frac{c^2}{x^2}.$$

Squaring,

$$4b^2x^2 - 4a^2b^2 = x^4 + 2b^2x^2 - 2c^2 + b^4 - \frac{2b^2c^2}{x^2} + \frac{c^4}{x^4}.$$

Adding $4c^2$ to each member,

$$x^4 - 2b^2x^2 + b^4 + 2c^2 - \frac{2b^2c^2}{x^2} + \frac{c^4}{x^4} = 4c^2 - 4a^2b^2.$$

Evolving, $x^2 - b^2 + \frac{c^2}{x^2} = 2\sqrt{c^2 - a^2b^2}$.†

Clearing of fractions, &c.,

$$x^4 - (b^2 + 2\sqrt{c^2 - a^2b^2})\,x^2 = -c^2,$$

$$x^4 - (b^2 + 2\sqrt{c^2 - a^2b^2})\,x^2 + (\tfrac{1}{2}b^2 + \sqrt{c^2 - a^2b^2})^2 =$$
$$\tfrac{1}{4}b^4 + b^2\sqrt{c^2 - a^2b^2} - a^2b^2.$$

Evolving,

$$x^2 - (\tfrac{1}{2}b^2 + \sqrt{c^2 - a^2b^2}) = \pm\, b\sqrt{\tfrac{1}{4}b^2 + \sqrt{c^2 - a^2b^2} - a^2}.$$

$$x = (\tfrac{1}{2}b^2 \pm \sqrt{c^2 - a^2b^2} \pm b\sqrt{\tfrac{1}{4}b^2 \pm \sqrt{c^2 - a^2b^2} - a^2})^{\frac{1}{2}} = \text{A C}.$$

$$\frac{c}{x} = (\tfrac{1}{2}b^2 \pm \sqrt{c^2 - a^2b^2} \mp b\sqrt{\tfrac{1}{4}b^2 \pm \sqrt{c^2 - a^2b^2} - a^2})^{\frac{1}{2}} = \text{A B}.$$

---

* Problem XXII. as printed in early editions of the Geometry, being considered too difficult for ordinary students, in the Sixth Edition it was changed.

† It is evident that c can never be less than $a\,b$. When $c = a\,b$, $\sqrt{c^2 - a^2b^2}$ becomes 0, and A is a right angle. The positive root renders the angle A acute, and appears to be always admissible; but the negative root makes A obtuse, and produces imaginary quantities, unless $b^2 > 4a^2 + 4\sqrt{c^2 - a^2b^2}$. The double sign has been supplied in the final results, though not used in the previous work. If $a = 9$, $b = 52$, and $c = 615$, the equation will have eight real roots, four positive and four negative.

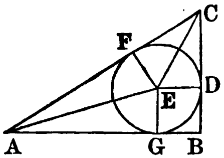
By adding $2c + b^2$ to each member of the equation $x^2 - b^2 + \frac{c^2}{x^2} = 2\sqrt{c^2 - a^2 b^2}$, extracting the square root, and reducing,

$$x = \frac{\pm\sqrt{b^2 + 2c \pm 2\sqrt{c^2 - a^2 b^2}} \pm \sqrt{b^2 - 2c \pm 2\sqrt{c^2 - a^2 b^2}}}{2}.$$

$$\frac{c}{x} = \frac{\pm\sqrt{b^2 + 2c \pm 2\sqrt{c^2 - a^2 b^2}} \mp \sqrt{b^2 - 2c \pm 2\sqrt{c^2 - a^2 b^2}}}{2}.$$

These values are equal to the preceding ones, though expressed in a different form. It will be observed that in both sets x and $\frac{c}{x}$ have each eight values.

Problem XXIII.



In the right-angled triangle A B C, let A C $= a$, radius of the inscribed circle $= b$, A B $= x$, and B C $= \sqrt{a^2 - x^2}$. Draw lines from A and C to E, the centre of the inscribed circle, and draw the radii E D, E G, E F to the points of contact. It is easily demonstrated that triangle A F E = triangle A G E, triangle E F C = triangle E D C, and E D B G = square of the radius; hence, A C = A G + D C, and A C + 2 E D = A B + B C, or,

$$a + 2b = x + \sqrt{a^2 - x^2}.$$

Transposing, $$a + 2b - x = \sqrt{a^2 - x^2}.$$

Squaring,

$$a^2 + 4ab + 4b^2 - 2ax - 4bx + x^2 = a^2 - x^2$$

Transposing, &c.,

$$x^2 - (a + 2b)x = -2ab - 2b^2.$$

Completing the square,

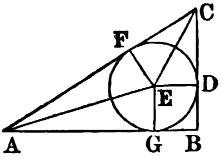
$$x^2 - (a + 2b)x + \left(\frac{a + 2b}{2}\right)^2 = \frac{a^2 - 4ab - 4b^2}{4}.$$

Evolving and transposing,

$$x = \frac{a + 2b \pm \sqrt{a^2 - 4ab - 4b^2}}{2} = \text{A B}.$$

$$\sqrt{a^2 - x^2} = \frac{a + 2b \mp \sqrt{a^2 - 4ab - 4b^2}}{2} = \text{B C}.$$

PROBLEM XXIV.



Construct the figure as in the last problem. Put $AC = a$, $AB - ED = AG = b$, $AB = x$, and $BC = \sqrt{a^2 - x^2}$. As in the last problem, $AC + 2ED = AB + BC$; $2ED = 2(AB - AG) = 2(x - b)$; hence, $a + 2x - 2b = x + \sqrt{a^2 - x^2}$.

Transposing, $a + x - 2b = \sqrt{a^2 - x^2}$.

Squaring,

$$a^2 + 2ax + x^2 - 4ab - 4bx + 4b^2 = a^2 - x^2.$$

Transposing, &c.,

$$x^2 - (2b - a)x = 2ab - 2b^2.$$

Completing the square,

$$x^2 - (2b - a)x + \left(\frac{2b - a}{2}\right)^2 = \frac{a^2 + 4ab - 4b^2}{4}.$$

Evolving and transposing,

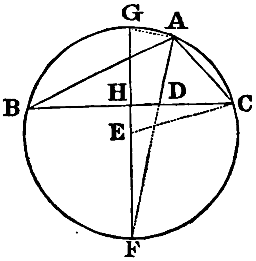
$$x = \frac{2b - a \pm \sqrt{a^2 + 4ab - 4b^2}}{2} = AB.$$

$$a + x - 2b = \sqrt{a^2 - x^2} = \frac{a - 2b \pm \sqrt{a^2 + 4ab - 4b^2}}{2} = BC.$$

It will be seen that the radicals in these values must have the same sign, while in the preceding, and many of the other problems, the radicals must have opposite signs. In this case, the signs of the rational quantities are changed. Were *b* the difference between B C and the radius, the above expressions would change places. This problem differs from the preceding in the fact that *b* bears the given relation to only one side of the triangle, instead of bearing a similar relation to each side.

PROBLEM XXV.

Let A B C be the triangle, A D the line bisecting the vertical angle, and B G A C F the circumscribing circle, having its centre at E. Draw the diameter G F perpendicular to the base B C, at its middle point, H. Extend A D, and it will meet both the circumference and diameter at the point F, as A F and G F each



bisect the arc B F C. Also draw E C and A G. Put B C $= b$, A D $= a$, G F $= d$, A B $= x$, and A C $= y$. H E $= \sqrt{\overline{EC}^2 - \overline{CH}^2} = \pm \frac{1}{2}\sqrt{d^2 - b^2}$ ∴ F H $= \frac{d \pm \sqrt{d^2 - b^2}}{2}$, the sign of the radical being positive when E is without the triangle, and negative when it is within, or, in other words, positive when the angle B A C is obtuse, negative when it is acute, and disappearing when it is a right angle. The triangles F H D and F A G are similar, because they are right-angled at H and A, and have the common angle F; hence F H : F D :: F A : F G. Substituting the values of these lines and putting $z =$ F D,

$$\frac{d \pm \sqrt{d^2 - b^2}}{2} : z :: z + a : d,$$

or,

$$z^2 + a z = \frac{d^2 \pm d\sqrt{d^2 - b^2}}{2}.$$

Completing the square,

$$z^2 + a z + \frac{a^2}{4} = \frac{a^2 + 2d^2 \pm 2d\sqrt{d^2 - b^2}}{4}.$$

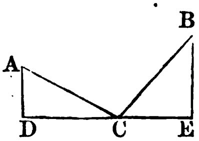
Evolving and transposing,

$$z = \frac{-a \pm (a^2 + 2d^2 \pm 2d\sqrt{d^2 - b^2})^{\frac{1}{2}}}{2} = \text{F D}.$$

B D × D C = F D × A D (Prop. XXXIII. Cor. Bk. IV.); hence, putting F D $= m$, B D $= u$, and D C $= b - u$, $u^2 - b u = -a m$, which, solved by the rules for quadratics, gives $u = \frac{b \pm \sqrt{b^2 - 4am}}{2} =$ B D, and $b - u = \frac{b \mp \sqrt{b^2 - 4am}}{2} =$ D C. A B × A C $= \overline{AD}^2 +$ B D × D C (Prop. XXXVI. Bk. IV.); therefore, taking s for the segment B D and s' for the segment D C, $x y = a^2 + s s'$. But $x : y :: s : s'$ (Prop. XIX. Bk. IV.), and $y = \frac{s' x}{s}$. Substituting this value in the preceding equation, $s' x^2 = a^2 s + s^2 s'$, $x^2 = \frac{a^2 s}{s'} + s^2$, $x = s\sqrt{\frac{a^2}{s s'} + 1}$, and $y = s'\sqrt{\frac{a^2}{s s'} + 1}$.

Since $s s' = a m$, $x = s \sqrt{\frac{a}{m} + 1}$, and $y = s' \sqrt{\frac{a}{m} + 1}$, or $x = \left(\frac{b \pm \sqrt{b^2 - 4 a m}}{2}\right) \sqrt{\frac{a}{m} + 1}$, and $y = \left(\frac{b \mp \sqrt{b^2 - 4 a m}}{2}\right) \sqrt{\frac{a}{m} + 1}$. The last values might have been obtained without using s and s', but their use renders the operation more simple.

PROBLEM XXVI.



Let A D and B E be the two pillars, and C the position of the foot of the ladder; then $AD = 20$, $BE = 30$, and $DE = 60$. Put $DC = x$ and $CE = 60 - x$. Since $AC = BC$,

$$20^2 + x^2 = (60 - x)^2 + 30^2,$$

or, $$400 + x^2 = 3600 - 120\,x + x^2 + 900.$$

Cancelling, &c., $$120\,x = 4100,$$

and $$x = 34\tfrac{1}{6} = DC.$$

$$60 - x = 60 - 34\tfrac{1}{6} = 25\tfrac{5}{6} = CE.$$

$$\sqrt{(34\tfrac{1}{6})^2 + 20^2} = \sqrt{\frac{56425}{36}} = \frac{237.5395}{6} = 39.5899 = AC = BC.$$

PROBLEM XXVII.

Let $x =$ the breadth, $84 - x =$ the length, and $\frac{7x}{25} =$ the depth. $$x^2 + (84 - x)^2 = 60^2,$$

or, $$x^2 + 7056 - 168\,x + x^2 = 3600.$$

Collecting and reducing,

$$x^2 - 84\,x = -1728.$$

Completing the square,

$$x^2 - 84\,x + 42^2 = 36.$$

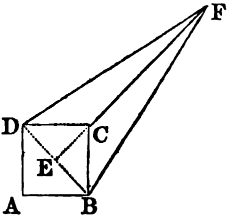
Evolving, $$x - 42 = \pm 6,$$

or, $$x = 36 = \text{breadth},$$

$$84 - x = 48 = \text{length},$$

and, $$\frac{7x}{25} = 10.08 = \text{depth}.$$

PROBLEM XXVIII.



Let ABCD be the square field, and F the position of the oak. Put $a = BF = DF = 78$, $b = CF = 59.161$, and $x = AB$. $BD = x\sqrt{2}$; $BE = \frac{1}{2}x\sqrt{2} = EC$; $EF = \frac{1}{2}x\sqrt{2} + b = \sqrt{a^2 - \frac{1}{2}x^2}$; therefore,

$$\tfrac{1}{2}x^2 + bx\sqrt{2} + b^2 = a^2 - \tfrac{1}{2}x^2,$$

and, $$x^2 + bx\sqrt{2} = a^2 - b^2.$$

Completing the square, $$x^2 + bx\sqrt{2} + \tfrac{1}{2}b^2 = a^2 - \tfrac{1}{2}b^2.$$

Extracting the root, $$x + \tfrac{1}{2}b\sqrt{2} = \pm\sqrt{a^2 - \tfrac{1}{2}b^2}.$$

$$x = \frac{-b \pm \sqrt{2a^2 - b^2}}{\sqrt{2}}.$$

Substituting the numerical values of a and b,

$$x = \frac{-59.161 + \sqrt{8668}}{\sqrt{2}} = \frac{93.102 - 59.161}{1.4142} = 24 = AB.$$

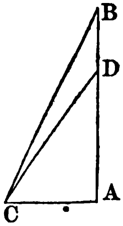
$$24^2 = 576\,\text{P.} = 3\,\text{A.}\ 2\,\text{R.}\ 16\,\text{P.}$$

The area expressed in letters is,

$$\left(\frac{-b + \sqrt{2a^2 - b^2}}{\sqrt{2}}\right)^2 = a^2 - b\sqrt{2a^2 - b^2},$$

which also gives 576.

PROBLEM XXIX.



Let AB represent the wall of the house, D the top of the second story, and C the position of the foot of the ladders. Put $x = CB$, and $x - 12 = CD$. As $AC = 20$, $AD = \sqrt{(x-12)^2 - 20^2}$, and $AB = \sqrt{x^2 - 20^2}$; hence

$$\sqrt{(x-12)^2 - 20^2} : \sqrt{x^2 - 20^2} :: 2 : 3,$$

or $$(x-12)^2 - 20^2 : x^2 - 20^2 :: 4 : 9,$$

and $$9(x-12)^2 - 3600 = 4x^2 - 1600.$$

From the last equation we obtain $5x^2 - 216x = 704$, which gives $x = 46.245$, the length of the longer ladder, CB. $AB = \sqrt{46.245^2 - 20^2} = \sqrt{1738.6} = 41.696$, the height of the house.

PROBLEM XXX.

The solidity of the small cone cut off to form the frustum $=$ $2513.28 - 2474.01 = 39.27$. $2513.28 \div 39.27 = 64$. If x and $x - 19.5$ represent the slant heights of the large and small cones, $x : x - 19.5 :: \sqrt[3]{2513.28} : \sqrt[3]{39.27} :: \sqrt[3]{64} : \sqrt[3]{1} :: 4 : 1$, and $x = 4x - 78$, from which $x = 26$. Put $y =$ altitude of cone, then $\sqrt{26^2 - y^2} =$ radius of the base, and $\frac{\pi}{3} y\,(26^2 - y^2)$ $=$ solidity of the cone $= 2513.28$,

or, $$-y^3 + 676\,y = \frac{2513.28}{1.0472} = 2400.$$

Multiplying by $-y$, $y^4 - 676\,y^2 = -2400\,y$.

Adding* $24^2 y^2 + 50^2$ to each member,

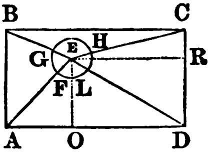
$$y^4 - 100\,y^2 + 50^2 = 24^2 y^2 - 2400\,y + 50^2.$$

Evolving, $$y^2 - 50 = 24\,y - 50,$$

and $$y^2 = 24\,y, \text{ or } y = 24 = \text{altitude.}$$

$$2\sqrt{26^2 - y^2} = 2\sqrt{100} = 20 = \text{diameter of base.}$$

PROBLEM XXXI.



Let A B C D represent the garden, and F G H L the fountain, of which E is the centre; then $AF = 40 = a$, $BG = 28 = b$, $CH = 52 = c$, $DL = 60 = d$, and $EF = x$. If perpendiculars were drawn from E, the sum of their squares would be equal to $\overline{AE}^2 + \overline{EC}^2$, and also equal to $\overline{BE}^2 + \overline{ED}^2$; hence, $\overline{AE}^2 + \overline{EC}^2 = \overline{BE}^2 + \overline{ED}^2$, or $(a + x)^2 + (c + x)^2$ $= (b + x)^2 + (d + x)^2$, which gives $2x = \frac{b^2 + d^2 - a^2 - c^2}{a + c - b - d} =$ $\frac{28^2 + 60^2 - 40^2 - 52^2}{40 + 52 - 28 - 60} = 20 =$ diameter of the fountain.

$AE = 40 + 10 = 50 = m$, $BE = 28 + 10 = 38 = n$, $CE = 52 + 10 = 62 = p$, and $DE = 60 + 10 = 70 = q$. Draw the perpendiculars E O, E R, and put $AD = y$, $DC = z$. $\overline{AE}^2 = \overline{DE}^2 + \overline{AD}^2 - 2\,OD \times AD$ (Prop. XII. Bk. IV.), or $m^2 = q^2 + y^2 - \text{O D} \times 2y$, and $\text{O D} = \frac{q^2 - m^2 + y^2}{2y}$. In like manner, $\text{D R} = \frac{q^2 - p^2 + z^2}{2z} = \text{E O}$. $\overline{\text{O D}}^2 + \overline{\text{E O}}^2 = \overline{\text{D E}}^2$, or $\left(\frac{q^2 - m^2 + y^2}{2y}\right)^2 + \left(\frac{q^2 - p^2 + z^2}{2z}\right)^2 = q^2$. $yz =$ area A B C D $= 1$ acre $= 4840$ sq. yds. $= \text{A}$; hence, $z = \frac{\text{A}}{y}$. Substituting this value of z in the preceding equation, and multiplying both sides by $4y^2$, $(q^2 - m^2 + y^2)^2 + \left(\frac{q^2 - p^2}{\text{A}} y^2 + \text{A}\right)^2 = 4q^2y^2$, or $(q^2 - m^2)^2 + 2(q^2 - m^2)y^2 + y^4 + \left(\frac{q^2 - p^2}{\text{A}}\right)^2 y^4 + 2(q^2 - p^2)y^2 + \text{A}^2 = 4q^2y^2$. Cancelling and arranging the terms, $\left[1 + \left(\frac{q^2 - p^2}{\text{A}}\right)^2\right] y^4 - 2(m^2 + p^2)y^2 = -\text{A}^2 - (q^2 - m^2)^2$. But $1 + \left(\frac{q^2 - p^2}{\text{A}}\right)^2 = 1 + \left(\frac{12}{55}\right)^2 = \frac{3169}{3025}$; $2(m^2 + p^2) = 2(6344)$; and $-\text{A}^2 - (q^2 - m^2)^2 = -23425600 - 2400^2 = -29185600$. Substituting these numbers in the above equation,

---

* $2400 = 24 \times 100$. See Greenleaf's Algebra, Art. 344.

$$\frac{3169}{3025} y^4 - 2(6344)y^2 = -29185600,$$

or,

$$y^4 - 2\left(\frac{19190600}{3169}\right) y^2 = -\frac{88286440000}{3169}.$$

Completing the square,

$$y^4 - 2\left(\frac{19190600}{3169}\right) y^2 + \left(\frac{19190600}{3169}\right)^2 = \frac{88499400000000}{3169^2}.$$

Evolving,

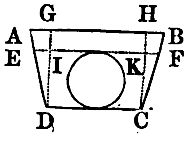
$$y^2 - \frac{19190600}{3169} = \pm \frac{9407412}{3169}, \text{ nearly.}$$

$$y^2 = \frac{19190600}{3169} + \frac{9407412}{3169} = 9024.3, \text{ nearly.}$$

$$y = \sqrt{9024.3} = 94.996 = \text{A D}.$$

$$z = \frac{4840}{94.996} = 50.949 = \text{D C}.$$

PROBLEM XXXII.

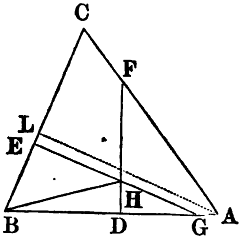


Let A B C D represent a vertical section of the vessel. Put $x = \text{D C}$ and $\frac{7x}{5} = \text{A B}$, the diameters of the two bases. The solidity of the vessel may be expressed thus, $\frac{\pi}{4}\left(x^2 + \frac{49x^2}{25} + \frac{7x^2}{5}\right) \times \frac{21}{3} = \frac{7\pi}{4}\left(\frac{109x^2}{25}\right) = \frac{763\pi x^2}{100}$; 8.67 ft. =

14981.76 in.; hence $\frac{768\,\pi\,x^2}{100} = 14981.76$, $\pi\,x^2 = 1963.5$, $x^2 = \frac{1963.5}{3.1416} = 625$, $x = 25 =$ D C, and $\frac{7\,x}{5} = 35 =$ A B.

The surface of any sphere $= \pi\,D^2$, and its solidity $= \frac{1}{6}\,\pi\,D^3$, then if $\frac{1}{6}\,\pi\,D^3 = 2\frac{1}{2} \times \pi\,D^2$, $D = 2\frac{1}{2} \times 6 = 15$; when $D = 15$, $\frac{1}{6}\,\pi\,D^3 = 1767.15 =$ solid contents of the globe. Drawing D G and C H perpendicular to A B, G H = D C = I K, and the triangles C K F and C H B are similar; whence C H : C K :: B H : F K, or $21 : 15 :: 5 : \frac{25}{7}$. E F $= 25 + 2 \times \frac{25}{7} = 32\frac{1}{7}$. The solidity of E F C D is $.785398\,(25^2 + 32\frac{1}{7}^2 + 25 \times 32\frac{1}{7}) \times \frac{15}{3} = 2461\,\frac{36}{49} \times 3.92699 = 9667.21$. $9667.21 - 1767.15 = 7900.06 =$ quantity of water, in inches. $7900 \div 231 = 34.2 =$ no. of gallons of water.

PROBLEM XXXIII.



Let the trees be placed at the points A, B, and C, of the triangle A B C. We first find the point D, on the line A B, equally distant from the tops of the trees A and B, as in Prob. XXVI. (See Prob. 34, Bk. XIII. for a geometrical construction.) Put B D $= x$, then $x^2 + 110^2 = (112 - x)^2 + 114^2$, and $224\,x = 13440$; hence $x = 60 =$ B D. To find E, a point on the line B C, equally distant from the tops of the trees B and C, put B E $= y$, and from the equation $y^2 + 110^2 = (104 - y)^2 + 98^2$, we obtain $208\,y = 8320$, or $y = 40 =$ B E. Draw D F perpendicular to A B, and E G perpendicular to B C. Any point of the line D F will be equally distant from the tops of the trees A and B,* while any point of the line E G will be equally distant from the tops of the trees B and C; hence H, the only point common to both lines, will be equally distant from the tops of the three trees.

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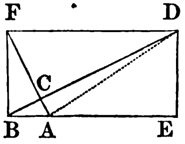
* As the plane A B C is perpendicular to the plane of the trees A and B, F D is perpendicular to the plane of the trees. (Prop. VIII. Bk. VII.) As D is equally distant from the tops of the two trees, any point of F D must also be equally distant from their tops. (Prop. V. Bk. VII.)

Draw A L perpendicular to B C, also draw B H. Put B L $= z$, then A L $= 112^2 - z^2 = 120^2 - (104 - z)^2$, $208z = 8960$, $z = 43\frac{1}{13}$ = B L. The triangles G B E and A B L are similar; hence B L : B E : : A B : G B, or $43\frac{1}{13} : 40 :: 112$: G B, and G B = 104. G B — B D = 104 — 60 = 44 = D G. $\sqrt{\overline{GB}^2 - \overline{BE}^2} = \sqrt{104^2 - 40^2} = \sqrt{9216} = 96$ = E G. The triangles B E G and H D G are similar; hence E G : D G : : B E : H D, or 96 : 44 : : 40 : H D, and H D $= 18\frac{1}{3}$. $\overline{BD}^2 + \overline{HD}^2 = 60^2 + (18\frac{1}{3})^2 = 3936\frac{1}{9} = \overline{BH}^2$. The length of the required line $= \sqrt{\overline{BH}^2 + 110^2} = \sqrt{3936\frac{1}{9} + 12100} = \sqrt{16036\frac{1}{9}} = 126.634$.

PROBLEM XXXIV.



Let D be the position of the oak-tree, D C the segment of the diagonal, 16 chains long, F A the perpendicular, of which the portion A C = 2 chains. Draw D A, and the triangles A B F and A B D are equivalent, as they have the same base, A B, and the same altitude, B F. Taking away the common portion, A B C, we have the remainders equivalent, or $\frac{1}{2}$ A C × C D = $\frac{1}{2}$ B C × C F. Putting $x =$ B C and $y =$ C F, $2 \times 16 = xy$. From the similar triangles A B C and B C F, we also obtain $2 : x :: x : y$, or $2y = x^2$. Substituting this value of y in the first equation, we find $x^3 = 64$, and $x = 4$ = B C; $2y = 16$, $y = 8$ = C F.

$\frac{4 \times 2}{2}$ = 4 chains = 1 R. 24 P. = area A B C.

$\frac{4 \times 8}{2}$ = 16 chains = 1 A. 2 R. 16 P. = area B C F.

$\frac{8 \times 16}{2}$ = 64 chains = 6 A. 1 R. 24 P. = area F C D.

B F $= \sqrt{4^2 + 8^2} = \sqrt{80} = 4\sqrt{5}$ = E D; F D $= \sqrt{8^2 + 16^2} = \sqrt{320} = 8\sqrt{5}$ = B E; $8\sqrt{5} \times 4\sqrt{5} = 160$ chains = 16 A. = area B E D F. 160 — (4 + 16 + 64) = 76 chains = 7 A. 2 R. 16 P. = area A E D C.

THE END.

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The NATIONAL ARITHMETIC is a thorough theoretical and practical treatise, containing a greater amount and variety of matter, strictly connected with the science, than can be found in any other treatise. As suited to the wants of the High Schools, Academies, Normal Schools, and Commercial Colleges, it has no equal, it is believed, in any language.

The ALGEBRA furnishes what has been hitherto much desired, a thorough practical and theoretical text-book, suited to the wants of elementary schools, as well as academies, in a single volume of a portable form. Very comprehensive in its plan and details, and progressive in its gradation of problems, it occupies the ground commonly given very inconveniently to two books.

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GREENLEAF'S MATHEMATICAL WORKS are no *untried* books, or of doubtful reputation. No other works of the kind have, in the same time, secured so general an introduction into the best schools in all parts of the United States, or have been as highly recommended by eminent teachers and mathematicians.

The Vermont Board of Education, in December, 1858, adopted this entire Arithmetical Series for the District Schools of that State ; and this selection, by law, has " become authoritative and binding upon the Board of Education, Superintendents and Teachers, until January 1, 1864 ;" or for five years.

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The National Arithmetic and Algebra are used as text-books in the four Massachusetts State Normal Schools, Connecticut State Normal School, and New Jersey State Normal School.

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W. H. Seavey, Principal of City Normal School, Boston.

Joseph Perry, formerly Professor in Dartmouth College, Hanover, N. H.

P. R. Lovejoy, Acting President of Newton University, Baltimore.

J. O'Neill, Principal of Prep. Dep. of St. Louis University, St. Louis.

D. W. Hoyt, Professor of Mathematics in Polytechnic College, Philadelphia.

Stiles French, Teacher of Classical and Scientific School, New Haven, Ct.

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David Patterson, M. D., Classical Teacher of City Normal School, and Principal of Public School No. 3, New York City.

D. N. Camp, formerly Professor of Mathematics in State Normal School, New Britain, Ct., and now State Superintendent of the schools of Connecticut.

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A. P. Stone, Principal of the High School, Plymouth, Mass., and President of Massachusetts State Teachers' Association.

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Merrick and Emory Lyon, Principals of University Grammar School, Providence.

George W. Gardner, Principal, and Ephraim Knight, Professor of Mathematics, Literary and Scientific Institution, New London, N. H.

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From the Principal of the Academy preparatory to Waterville College.

After a careful examination and comparison of several of the later works on Arithmetic, with a view to introduction, I selected Greenleaf's National Arithmetic in preference to any other; and, from subsequent acquaintance obtained by teaching it, I can say without hesitation that it has more than answered my expectations. While it is full of information needed by all classes, but difficult of access in any other form, its merits as a text-book can be fully appreciated only by the practical teacher, to whom it is an invaluable aid. In its choice of topics, its logical arrangement, the clearness and conciseness of its rules and definitions, in the accuracy of its tables and other matter, and especially in its practical applications, it far surpasses any other work of the kind with which I am acquainted.

J. S. HAMBLEN, *Principal of Waterville Academy.*

Waterville, Me., Dec. 28, 1858.

From the Teacher of Mathematics in N. Y. Conference Seminary, Charlotteville, N. Y.

After a careful examination of your late edition of the National Arithmetic, I fully appreciate the merits of the work, and esteem it as a valuable improvement in the science of Arithmetic. I have resolved to use it in preference to any other in my department in this Seminary, and I have already organized a class in the work. I feel grateful for the assistance Mr. Greenleaf has given, by the judicious management of such text-books as shall readily impart to the student the true end of the study of Arithmetic.

JOHN ROBERTS.

January 17, 1859.

From the Principal of the High School, Ware, Mass.

I have used in my school "Greenleaf's National Arithmetic," and his "Treatise on Algebra," and feel prepared to say that for my classes they are the best text-books I have seen. The explanations are lucid and to the point. The problems are such as interest the pupils, and excite a lively emulation among the members of the class.

The Geometry, by the same author, I have examined, and do not hesitate to recommend it in preference to any treatise on the subject with which I am acquainted. Its practical problems are a valuable addition to what we find in other similar treatises.

These works form a mathematical series for High Schools *eminently practical*, and peculiarly adapted to their wants.

GEO. S. GROSVENOR.

Ware, Mass., Jan. 29, 1859.

From the late Principal of Thomaston Academy, now Teacher of Select High School, Rockland, Me.

Having for many years used Greenleaf's Series of Arithmetics, I do, without hesitation, say, that, in my opinion, they are preferable to any other of the kind with which I am acquainted. The new improved editions I think are decidedly the best Arithmetics now in use, or the best that I have ever seen.

Rockland, Me., Jan. 25, 1859.

HENRY PAINE.

From the Principals of the Grammar Schools, Rockland, Me.

We have examined and used Greenleaf's improved editions of the Common School and National Arithmetics, and think them much better adapted to our schools than any others with which we are acquainted.

A. T. LOW, *Summer-street Grammar School.*
A. L. TYLER, *Pine Grove Grammar School.*
G. F. BURGESS, *Oak Grove Grammar School.*
J. T. YOUNG, *Crescent-street Grammar School.*

Rockland, Me., Jan. 24, 1859.

RECOMMENDATIONS OF GREENLEAF'S GEOMETRY.

This work has been prepared at the earnest solicitation of many eminent teachers, who, having used the author's Arithmetics and Algebra with satisfaction, have been desirous of seeing his series rendered more complete by the addition of a Treatise on Geometry.

In common with most compilers of the present day, he has followed, in the main, the simple and elegant order of arrangement adopted by Legendre ; but, in the methods of demonstration, no particular authority has been closely followed, the aim having been to adapt the work fully to the latest and most improved modes of instruction. In this respect, it is believed, there will be found incorporated a considerable number of important improvements.

More attention than is usual in elementary works of this kind has been given to the *converse* of propositions. In almost all cases where it was possible the converse of a proposition has been demonstrated.

The demonstration of Proposition XX. of the first book is essentially the one given by M. da Cunha in the *Principes Mathématiques*, which has justly been pronounced by the highest mathematical authorities to be a very important improvement in elementary geometry. It has, however, never before been introduced into a text-book by an American author.

The Applications of Geometry to Mensuration, given in the eleventh and twelfth books, are designed to show how the theoretical principles of the science are connected with manifold practical results.

The Miscellaneous Geometrical Exercises which follow are calculated to test the thoroughness of the scholar's geometrical knowledge, besides being especially adapted to develop skill and discrimination in the demonstration of theorems and the solution of problems unaided except by principles.

Sufficient Applications of Algebra to Geometry are given to show the relation existing between these two branches of the mathematics. The problems introduced in connection therewith will be found to be not only of a highly interesting character, but well calculated to secure valuable mental discipline.—*Preface.*

We subjoin the following testimonials, as expressive of the opinion entertained by practical teachers, who have given the work a critical examination.

From A. M. GAY, A. M., *Principal of the High School, Charlestown, Mass.*

CHARLESTOWN, July 2, 1858.

From an examination, and a practical test in some of my classes, of the advanced sheets of Greenleaf's Geometry, I have no hesitation in pronouncing it superior to any class-book of the kind with which I am acquainted. The basis of the work is Legendre's Elementary Geometry, which has been long and favorably known to teachers. Much valuable matter, however, has been added by the present editor, so that it cannot be regarded as a translation, or a mere compilation. The language in which the propositions are stated is remarkably clear and concise. The development of principles is sufficiently extended, and in all respects it seems a book well adapted to the purposes of instruction in Academies, High Schools, and other institutions of learning. A. M. GAY.

From JONATHAN MARSHALL, A. M., *Principal of Northampton High School.*

NORTHAMPTON, MASS., Aug. 4, 1858.

From a careful examination of Greenleaf's Geometry I regard it as well adapted to the wants of schools and seminaries. It contains many valuable problems and propositions not found in other works. Aside from the subject-matter, the typography and mechanical execution of the book bespeak an excellence not surpassed by any other book of the kind. I shall undoubtedly introduce the work into my school. JONATHAN MARSHALL.

From D. B. HAGAR, A. M., *President of the Massachusetts Teachers' Association.*

WEST ROXBURY, MASS., July 3, 1858.

Having carefully read the advanced sheets of Greenleaf's Geometry, I am happy to be able to assure you that the work appears to me to be one of great merit. Of course no person can expect to find much that is essentially new in a geometrical treatise published at this day. But Mr. Greenleaf's, while it is based, like nearly all American Geometries, upon Legendre's original work, contains valuable propositions not usually found in similar publications, and also presents uncommonly clear demonstrations of many of the standard propositions. The definitions contained in this work are generally concise, clear, and exact. The exercises in Mensuration, and in the Application of Algebra to Geometry, will be found unusually interesting and useful. In a word, I have no doubt that Mr. Greenleaf's Geometry will be found to compare favorably with the best Geometries yet published.

D. B. HAGAR,
Principal of Eliot High School.

From A. P. STONE, A. M., *Principal of the High School, Plymouth, Mass.*

PLYMOUTH, July 20, 1858.

I have examined, with considerable care, Greenleaf's Geometry, and am prepared to give it a decided preference to any other text-book in that branch with which I am acquainted. I was, some time since, furnished with advanced sheets, which I compared with other text-books in use in my classes. The arrangement is good, the definitions clear and concise, and the language used in the demonstration of the propositions is lucid in a high degree. That portion of the book devoted to Mensuration, and to Miscellaneous Geometrical Exercises, is a highly valuable feature, and is well executed. It will furnish the teacher with the means of a thorough practical review of the whole subject. As regards the mechanical part of the book, the typography, diagrams, paper and binding, will be found superior to other similar works. I shall introduce this Geometry into my school next term.

A. P. STONE.

NEW SALEM (MASS.) ACADEMY, Aug. 12, 1858.

I have examined, with much pleasure, "Greenleaf's Elements of Geometry," and think it, in most respects, superior to any text-book of the kind now in use. The author seems to have attained the great object in all mathematical treatises, and especially in a Geometry, — clearness of demonstration united with conciseness of expression. He has closely followed the principle laid down in his Arithmetics and Algebra, — not to explain *too much*, nor *too little ;* and thus has made Geometry the study that it ought to be, — interesting as well as useful.

The Problems in Books XI.— XIV. are well selected, and cannot fail to put the theoretical knowledge of the student to a practical test.

J. A. SHAW, *Principal.*

From MOSES WOOLSON, A. M., *Principal of Girls' High School, Portland, Me.*

PORTLAND, July 31, 1858.

Upon a cursory examination of Greenleaf's Geometry, I find, although Legendre's admirable text-book has been taken, in a measure, as a basis, it evidently excels that work, in many points, in clearness and precision. Both in the matter of demonstration, and in the arrangement, important improvements are noticeable. Much extraneous material, too often found in editions of Legendre, is judiciously left out in this new book. I am much pleased with the work, and would therefore cheerfully commend it to the favorable consideration of teachers.

MOSES WOOLSON.

UXBRIDGE, MASS., Dec. 3, 1856.

I have used Greenleaf's Algebra two years. I was fully persuaded by previous careful examination that it was superior to any other I had used or examined.

The principles are clearly demonstrated, happily arranged, and applied to practical questions in the masterly manner peculiar to the author of Greenleaf's Series. My beginners have advanced in it more successfully than in any other text-book, and it has taxed to the highest degree the advanced and best scholars.

Its merits have exceeded my expectations, and I know of no work on Algebra that can supply its place in our High School. HENRY R. PIERCE,

Principal of the High School, Uxbridge, Mass.

From the PRINCIPAL OF MASSACHUSETTS STATE NORMAL SCHOOL, *Bridgewater, Mass.*

STATE NORMAL SCHOOL, BRIDGEWATER, Jan. 8, 1858.

We continue, in this institution, to make daily use of Greenleaf's National Arithmetic, regarding it, on the whole, a sterling work. Having recently examined the revised edition of the same book, I consider it, in many respects, an improvement on the former edition, and a work destined to be extensively used.

MARSHALL CONANT.

From A. PARISH, A. M., *Principal of the High School, Springfield, Mass., and one of the Massachusetts Board of Education.*

SPRINGFIELD, October 14, 1857.

I think, from the trial we have given Greenleaf's New National Arithmetic, it works admirably. I do not think any Arithmetic with which I am acquainted will make so strong arithmeticians as that work. ARIEL PARISH.

From the PRINCIPAL OF THE BOSTON CITY NORMAL SCHOOL.

GIRLS' HIGH AND NORMAL SCHOOL, BOSTON, June 3, 1858.

I am very much pleased with the new edition of the National Arithmetic, which has been used in this school since last September. I consider it a great improvement even on the previous edition of the same work, and this had proved a highly acceptable text-book in the schools till the introduction of the present edition.

W. H. SEAVEY.

From the PRESIDENT OF THE MASSACHUSETTS STATE TEACHERS' ASSOCIATION.

ELIOT HIGH SCHOOL, WEST ROXBURY, Oct. 2, 1857.

The critical examination which I have given to the revised edition of Greenleaf's "National Arithmetic" has satisfied me that it is a very great improvement upon former editions of that work. Rewritten throughout, evidently with much care, it embraces all the valuable features of its predecessor, omitting whatever experience had shown to be of little importance, while, at the same time, it presents a large amount of new and useful matter. Improvement is manifested chiefly in a fuller discussion of the several subjects, in more concise and accurate rules, and in more rigid analyses of numerous problems. The important additions and improvements which the author, in his preface, claims to have made, will, upon examination, be manifest to every one. Both author and publishers may, I am sure, congratulate themselves upon having prepared a work which cannot fail to command the public favor. D. B. HAGAR, *Principal.*

WARREN (R. I.) LADIES' SEMINARY, Aug. 11, 1857.

Greenleaf's Series of Arithmetics have been constantly used in this Seminary the past ten years, and now, after careful comparison, we see no reason to displace them for any other work on practical Arithmetic.

Greenleaf's Algebra, also, has been recently adopted in our classes, as combining advantages superior to those of any other treatise we have used. Most cordially do we recommend the above-named works for their accuracy of statement and variety of illustrations of mathematical principles.

A. M. GAMMELL, *Principal.*

From the NEW HAMPSHIRE BOARD OF EDUCATION.

HANOVER, N. H., May 31, 1858.

I would hereby certify that the Board of Education for the State of New Hampshire, at their late meeting in April, voted to recommend Greenleaf's Series of Arithmetics, in connection with Adams' and Burnham's, for use in the common schools of the State. J W. PATTERSON,
Secretary of the Board of Education.

I have regarded Greenleaf's Arithmetics as among the best in use. The recent additions which he has made render them additionally valuable. I cheerfully concur with Dr. Patterson as to their superior excellence. W. J. O'DONNELL.

POMFRET, VT., Aug. 18, 1856.

Having thoroughly examined Greenleaf's Arithmetics, and having used them much for the last six years as school text-books, and, for a part of the time, in connection with other works, I am fully satisfied of their superior merits. The author has very happily introduced all the important principles of Arithmetic, and has illustrated those principles by clear and comprehensive rules, followed by copious sets of examples that are well calculated to test and develop the mental powers of the scholar. I shall continue to recommend them to teachers and scholars, in preference to any other works with which I am acquainted.

His Common School Arithmetic is a model of excellence.

HOSEA DOTEN,
Town Superintendent of Schools, and Principal of Pomfret High School and Teachers' Institute.

BARRE (VT.) ACADEMY, July 5, 1857.

I have used Greenleaf's Common School Arithmetic for several years, and during the same time I have had many others placed in my hands for examination; but, so far as I am able to judge, I have never seen an intermediate Arithmetic that is its equal; and I would recommend its use in those schools in which intellectual development is the primary object.

I would also say the same in regard to Greenleaf's Algebra, which is still used in my school with satisfaction. J. S. SPAULDING, *Principal.*

From A. H. BUCK, A. M., *Principal of Roxbury Latin School.*

ROXBURY, MASS., Jan. 20, 1858.

You ask my opinion of Greenleaf's Arithmetics. In reply, I shall affirm their general superiority to any other series extant. I find the excellence of the "National Arithmetic" to be admitted by all conversant with books of this sort.

This truly deserves the name of "Higher Arithmetic," being a Thesaurus of the science of numbers, as well as a convenient and practical text-book.

I am, however, more immediately acquainted with Greenleaf's "Introduction to the National Arithmetic," having, till the beginning of the present school year, used this only in the Latin School. I can, with a confidence based on experience, assert the merits of this book, and on the following grounds:

1. Its arrangement is sufficiently symmetrical, inductive, and logical.
2. Its definitions, rules, explanations, and demonstrations, are clear, exact, and concise.
3. Its examples are, in my opinion, better than those of any other work. They are copious, various, and practical.
4. The Tables of Money, Weights, Measures, &c., are very complete and accurate, and are accompanied by suitable notes and explanations.
5. The frequent introduction of miscellaneous examples furnishes an excellent test of the scholar's proficiency and capacity to employ his own resources and acquisitions.
6. The work is complete in itself, and yet serves well as a means of preparation for the use of the "National." A. H. BUCK.

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